

# Influence of counter-rotating von Kármán flow on cylindrical Rayleigh-Bénard convection

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The axisymmetric flow in an aspect-ratio-one cylinder whose upper and lower bounding disks are maintained at different temperatures and rotate at equal and opposite velocities is investigated. In this combined Rayleigh-Bénard/von Kármán problem, the imposed temperature gradient is measured by the Rayleigh number  $Ra$  and the angular velocity by the Reynolds number  $Re$ . Although fluid motion is present as soon as  $Re \neq 0$ , a symmetry-breaking transition analogous to the onset of convection takes place at a finite Rayleigh number higher than that for  $Re = 0$ . For  $Re < 95$ , the transition is a pitchfork bifurcation to a pair of steady states, while for  $Re > 95$ , it is a Hopf bifurcation to a limit cycle. The steady states and limit cycle are connected via a pair of SNIPER bifurcations except very near the Takens-Bogdanov codimension-two point, where the scenario includes global bifurcations. Detailed phase portraits and bifurcation diagrams are presented, as well as the evolution of the leading part of the spectrum, over the parameter ranges  $0 \leq Re \leq 120$  and  $0 \leq Ra \leq 30\,000$ .

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## I. INTRODUCTION

Thermal convection and shear are central to the study of hydrodynamic instabilities. The interest in these stems both from their large number of practical applications and also from their status as prototypes in theoretical investigation. In this study, we combine two well-known cylindrical configurations: Rayleigh-Bénard convection in which the upper and lower bounding disks of the cylinder are maintained at different temperatures, and von Kármán flow in which the disks rotate at equal and opposite velocities. We choose the simplest small-aspect-ratio geometry: a cylinder with equal height and radius, with imposed axisymmetry.

There is an extensive literature on Rayleigh-Bénard convection in a small to medium aspect-ratio cylinder (in which the radius is between 0.5 and 5 times the height), e.g. [1–6]. The references most relevant to us are axisymmetric simulations [7–9] which produced patterns of radially travelling concentric rolls (target patterns). These were shown to be produced in most cases by a saddle-node infinite period (SNIPER) bifurcation which will be the subject of much of the present investigation.

The literature on von Kármán flow is not as voluminous, but is growing. Restricting ourselves to the small

to medium aspect-ratio and the exactly or nearly exactly counter-rotating configuration, recent numerical articles include axisymmetric and non-axisymmetric simulations [10–16]. Interest in this problem has been enhanced by experiments using this configuration at high Reynolds numbers to generate turbulence [17, 18] or a magnetic field via the dynamo effect [19–21].

The combined Rayleigh-Bénard/von Kármán configuration seems, however, unexplored. This situation stands in contrast to that of other shear and rotating flows. The superposition of plane Poiseuille flow with Rayleigh-Bénard convection (PRB), has been the subject of a great many investigations; indeed, a recent review [22] cites hundreds of articles. For longitudinal rolls in plane Poiseuille and Couette flows, the threshold remains unchanged [23, 24]. For rolls transverse to the Poiseuille flow, the threshold increases and its nature changes from absolute to convective, e.g. [25]. In Taylor-Couette flow between cylinders at different temperatures, the nature of the instability also depends strongly on the angle between the imposed shear and buoyancy force, e.g. [26, 27]. Turning to rotating disks, heat transfer in the rotor-stator configuration (between a rotating and a stationary disk) is extensively studied because of its applications to turbomachinery, e.g. [28]. Rotating Rayleigh-Bénard convection has long interested geophysicists, as well as researchers in pattern formation e.g. [29], because of the possibility of chaos at onset; SNIPER bifurcations are found in this system as well [30, 31]. Of the studies we

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have found of the flow between counter-rotating disks maintained at different temperatures, one computes the similarity solution between disks of infinite radius prior to the onset of convection [32], and another considers turbulent flows and neglects buoyancy [33].

The Rayleigh-Bénard/von Kármán problem in this geometry turns out to display a number of fascinating phenomena. For this reason, this problem has been used to develop and test a reduced model for numerical simulation, in which fields are expanded in terms of the eigenfunctions of the governing equations linearized about a steady state; this study is the subject of a companion paper [34].

Our paper is organized as follows. Section II presents the governing equations of our configuration, along with the non-dimensionalizations we have used. In section III, we describe the various codes we have used to carry out time-dependent simulations, branch continuation and linear stability analysis. Section IV shows how the transition threshold is affected by the disk counter-rotation. In sections V and VI, we present the steady states and limit cycles resulting from the transition. The connection between the steady states and limit cycle via a SNIPER bifurcation is explained in section VII, which presents a complete phase diagram and bifurcation analysis. Reversing the traditional order, section VIII gives a linear stability analysis of the basic state, showing how the convective eigenvalues and eigenvectors are interlaced and joined by the von Kármán flow.

## II. PROBLEM FORMULATION

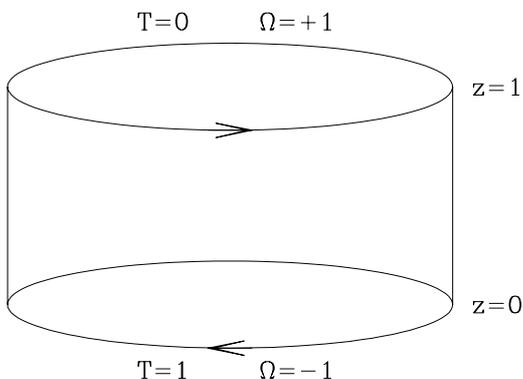


FIG. 1: Cylinder with height and radius equal to one. The bottom disk is heated and rotated in the clockwise direction, while the top disk is cooled and rotated in the counter-clockwise direction.

We consider a viscous Newtonian fluid governed by the Boussinesq approximation and contained in a cylinder of radius  $R$  and height  $H$ . The velocity and temperature fields are assumed axisymmetric. The bounding cylinder is stationary and thermally insulating. The upper

and lower bounding disks are thermally conducting, rotate with angular velocities  $+\Omega$  and  $-\Omega$  and are held at temperatures  $T_0$  and  $T_0 + \Delta T$ , respectively. The nondimensional parameters are defined as:

$$Re = \frac{\Omega R^2}{\nu}, \quad Ra = \frac{g\gamma\Delta T R^3}{\alpha\nu}, \quad Pr = \frac{\nu}{\alpha}, \quad \Gamma = \frac{R}{H} \quad (1)$$

where  $\nu$ ,  $\alpha$ ,  $\gamma$  and  $g$  are the kinematic viscosity, the thermal diffusivity, the thermal expansion coefficient, and the gravitational acceleration, respectively. The Prandtl number  $Pr$  and aspect ratio  $\Gamma$  are set to one. Reynolds numbers in the range  $0 \leq Re \leq 120$  will be studied, along with Rayleigh numbers in the range  $0 \leq Ra \leq 30000$ .

Our study uses several independent codes, summarized in section III, which use different non-dimensionalizations, and different problem formulations, i.e. primitive variables or streamfunction-vorticity. To facilitate later discussion, we briefly present these various formulations here. The non-dimensionalized temperature is taken to be the deviation from  $T_0$  divided by  $\Delta T$  and lengths are non-dimensionalized by  $R$ . The imposed counter-rotation of the disks provides a natural timescale: if time is non-dimensionalized by  $1/\Omega$  and velocities by  $R\Omega$ , the resulting governing equations are:

$$\nabla \cdot \mathbf{U} = 0 \quad (2a)$$

$$\partial_t \mathbf{U} + (\mathbf{U} \cdot \nabla) \mathbf{U} = -\nabla P + \frac{1}{Re} \Delta \mathbf{U} + \frac{Ra}{Pr Re^2} T \mathbf{e}_z \quad (2b)$$

$$\partial_t T + \mathbf{U} \cdot \nabla T = \frac{1}{Pr Re} \nabla^2 T \quad (2c)$$

where  $\Delta$  and  $\nabla^2$  denote the vector and scalar Laplacian, respectively. The boundary conditions are:

$$U_r = U_z = 0, \quad U_\theta = +r, \quad T = 0 \quad \text{at } z = 1 \quad (3a)$$

$$U_r = U_z = 0, \quad U_\theta = -r, \quad T = 1 \quad \text{at } z = 0 \quad (3b)$$

$$U_r = U_z = 0, \quad U_\theta = 0, \quad \partial_r T = 0 \quad \text{at } r = 1 \quad (3c)$$

$$U_r = \partial_r U_z = 0, \quad U_\theta = 0, \quad \partial_r T = 0 \quad \text{at } r = 0 \quad (3d)$$

These equations cannot be used in the absence of rotation, when  $Re = 0$ . To allow for this case, the unit of time can instead be taken to be the viscous diffusion time  $R^2/\nu$ , with the resulting velocity scale  $\nu/R$ . In addition, with the assumption of axisymmetry, the meridional velocity components ( $V_r, V_z$ ) are best described by the Stokes streamfunction  $\Psi$ :

$$V_r \mathbf{e}_r + V_z \mathbf{e}_z = \frac{\mathbf{e}_r}{r} \partial_z \Psi - \frac{\mathbf{e}_z}{r} \partial_r \Psi = \frac{\mathbf{e}_\theta}{r} \times \nabla \Psi \quad (4a)$$

Defining operators

$$\partial_\pm \equiv \partial_r \pm \frac{1}{r}, \quad (4b)$$

the azimuthal vorticity is:

$$D^2 \Psi \equiv \frac{1}{r} \left( \nabla^2 - \frac{2}{r} \partial_r \right) \Psi = \frac{1}{r} (\partial_- \partial_r + \partial_z^2) \Psi \quad (4c)$$

and the Navier-Stokes and Boussinesq equations reduce to:

$$\partial_s D^2 \Psi + (V_r \partial_- + V_z \partial_z) D^2 \Psi = \left( \nabla^2 - \frac{1}{r^2} \right) D^2 \Psi - \frac{Ra}{Pr} \partial_r T + \partial_z \left( \frac{V_\theta^2}{r} \right) \quad (5a)$$

$$\partial_s T + (V_r \partial_r + V_z \partial_z) T = \frac{1}{Pr} \nabla^2 T \quad (5b)$$

$$\partial_s V_\theta + (V_r \partial_r + V_z \partial_z) V_\theta = \left( \nabla^2 - \frac{1}{r^2} \right) V_\theta \quad (5c)$$

The rotation then appears in the boundary conditions, which become:

$$\Psi = \partial_z \Psi = 0 \iff V_r = V_z = 0, \quad V_\theta = +rRe, \quad T = 0 \quad \text{at } z = 1 \quad (6a)$$

$$\Psi = \partial_z \Psi = 0 \iff V_r = V_z = 0, \quad V_\theta = -rRe, \quad T = 1 \quad \text{at } z = 0 \quad (6b)$$

$$\Psi = \partial_r \Psi = 0 \iff V_r = V_z = 0, \quad V_\theta = 0, \quad \partial_r T = 0 \quad \text{at } r = 1 \quad (6c)$$

$$\Psi = D^2 \Psi = 0 \iff V_r = \partial_r V_z = 0, \quad V_\theta = 0, \quad \partial_r T = 0 \quad \text{at } r = 0 \quad (6d)$$

The velocity and time in (2)-(3) are related to those in (5)-(6) by  $\mathbf{V} = Re \mathbf{U}$  and  $s = t/Re$ ; the temperature  $T$  and all lengths remain unchanged. Equations (5c) and (5b) can be considered to be advection-diffusion equations for the azimuthal velocity  $V_\theta$  and the temperature  $T$ . Equation (5a) shows that gradients in  $V_\theta$  and  $T$  in turn generate vorticity. The vertical gradient  $\partial_z (V_\theta^2/r)$ , when combined with the boundary conditions (6a)-(6b), corresponds to Ekman pumping, by which the  $z$ -dependent  $V_\theta$  generates meridional vorticity and velocity for any non-zero  $Re$ . This meridional velocity in turn affects the azimuthal velocity and the temperature via equations (5c) and (5b).” However, the thermal gradient term  $\partial_r T$  in the vorticity equation (5a) is radial, and so the axial temperature gradient imposed by boundary conditions in (6a)-(6b) does not immediately impart motion to the fluid for any non-zero  $Ra$ . That is, Rayleigh-Bénard convection occurs via an instability at a finite threshold  $Ra_c$ , in contrast to natural convection, in which an externally imposed horizontal gradient generates motion for any  $Ra > 0$  via the term  $Ra \partial_r T$ .

Finally, we state the symmetries of the configuration of our problem, which can be seen from figure 1. The Rayleigh-Bénard convection problem has the symmetry of combined reflection in  $z$  and in  $T$  about their mean values, sometimes called the Boussinesq symmetry. The von Kármán flow has the symmetry of combined reflection in  $z$  and in  $\theta$ , called  $R_\pi$  [15]. (In this axisymmetric problem, reflection in  $\theta$  means only reversing the sign of  $U_\theta$ ; since all solutions are invariant under rotations in  $\theta$ , these are not considered.) The Rayleigh-Bénard/von Kármán problem thus has the symmetry which combines reflection in  $z$ ,  $\theta$  and  $T$ . A state is reflection-symmetric

if it is invariant under the reflection operator defined in the two formulations by:

$$\kappa : \begin{pmatrix} U_r \\ U_\theta \\ U_z \\ T \end{pmatrix} (r, z) \equiv \begin{pmatrix} U_r \\ -U_\theta \\ -U_z \\ 1 - T \end{pmatrix} (r, 1 - z) \quad \text{or} \\ \kappa : \begin{pmatrix} \Psi \\ U_\theta \\ T \end{pmatrix} (r, z) \equiv \begin{pmatrix} -\Psi \\ -U_\theta \\ 1 - T \end{pmatrix} (r, 1 - z) \quad (7)$$

### III. NUMERICAL METHODS

We have solved the governing equations using a number of different codes. The first is the time-integration code written by Patankar [35] for equations (2)-(3). The temporal discretization uses the backwards differentiation formula for the time derivative, the explicit Adams-Bashforth formula for the advective and buoyancy terms, and implicit evaluation of the diffusive terms, which leads to the following form for timestepping equation (2):

$$\frac{1}{2\Delta t} \left[ 3 \begin{pmatrix} \mathbf{U} \\ T \end{pmatrix}^{t+\Delta t} - 4 \begin{pmatrix} \mathbf{U} \\ T \end{pmatrix}^t + \begin{pmatrix} \mathbf{U} \\ T \end{pmatrix}^{t-\Delta t} \right] + 2 \begin{pmatrix} (\mathbf{U} \cdot \nabla) \mathbf{U} \\ \mathbf{U} \cdot \nabla T \end{pmatrix}^t - \begin{pmatrix} (\mathbf{U} \cdot \nabla) \mathbf{U} \\ \mathbf{U} \cdot \nabla T \end{pmatrix}^{t-\Delta t} = \begin{pmatrix} \frac{1}{Pr} \Delta \mathbf{U} \\ \frac{1}{Pr} \nabla^2 T \end{pmatrix}^{t+\Delta t} - \begin{pmatrix} \nabla P \\ 0 \end{pmatrix}^{t+\Delta t} + 2 \begin{pmatrix} \frac{Ra}{Pr Re^2} T \mathbf{e}_z \\ 0 \end{pmatrix}^t - \begin{pmatrix} \frac{Ra}{Pr Re^2} T \mathbf{e}_z \\ 0 \end{pmatrix}^{t-\Delta t}$$

The spatial discretization uses centered finite differences. A staggered grid is used, with pressure and temperature represented on the nodes and the velocity components between the nodes. A timestep  $\Delta t = 10^{-3}$  and a resolution in the radial and axial direction of  $82 \times 82$  were used.

Time integration has also been carried out with a second code, which uses the streamfunction-vorticity formulation (5)-(6), a second-order finite difference scheme in space, and an alternating-direction-implicit scheme in time [36]. A  $51 \times 51$  grid with a typical timestep of  $\Delta s = 10^{-4}$  were found to be adequate for obtaining converged results.

We also carried out bifurcation and linear stability analysis. We followed branches via Newton’s method and calculated eigenvalues via ARPACK. Several different spatial representations have been used. One code uses the same spatial discretization – i.e. the streamfunction-vorticity formulation and the same spatial resolution – as the time-integration code mentioned in the previous paragraph. The Jacobian of the matrix built in the last step of Newton iteration is directly used in ARPACK in shift-invert mode. The branches are followed using an

arclength continuation as described in [37] and the practical implementation of bordered matrix inversion can be found in [38].

The second code [39] with which we carried out bifurcation and linear stability analysis uses the primitive variable formulation and a pseudo-spectral spatial discretization. The number of Chebyshev polynomials (or Gauss-Lobatto collocation points) used is 15 in each of the  $r$  and  $z$  directions.

The quantitative results which we present have been obtained by two or more of the codes described above, and have also been verified by varying the resolution.

#### IV. THRESHOLDS

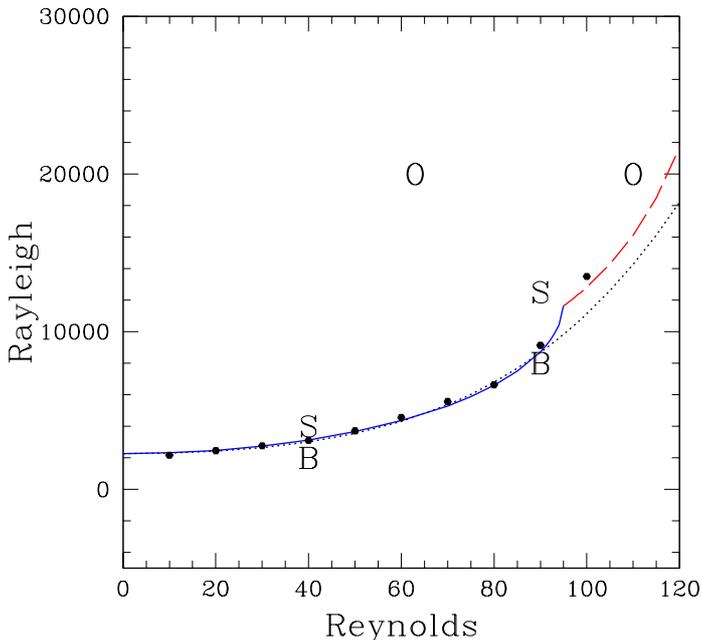


FIG. 2: (Color online) Convection thresholds as a function of Reynolds number. Solid blue curve: transition to steady convection. Dashed red curve: transition to oscillatory convection. Dotted black curve: fit to even fourth-order polynomial (9). Points: thresholds obtained from time-dependent code. Letters indicate parameter values for flows presented in figures 3 and 4 (B, basic state), 5 and 6 (S, steady convection), and 7 and 8 (O, oscillatory convection).

In the absence of rotation, i.e.  $Re = 0$ , the governing equations have as a solution the conductive state, in which the fluid is stationary and the temperature varies linearly in the vertical direction:

$$\mathbf{U} = 0, \quad T = 1 - z \quad (8)$$

Although the conductive solution exists for all  $Ra$ , it loses stability at some critical value  $Ra_c$  to a convective solution, in which fluid motion ensues, breaking the Boussinesq reflection symmetry (7). In the von Kármán system,

as mentioned in section II, Ekman pumping leads to recirculating regions for any non-zero value of  $Re$ . However, for a fixed value of  $Re$ , there is still a transition at a well-defined  $Ra_c$ , at which one solution loses stability to another. By analogy with the non-rotating case, we will call these the *basic* and *convective* solutions, despite the presence of fluid motion (and thus of convective heat transfer) for  $Ra < Ra_c$ , and we will continue to refer to the transition between them as the onset of convection.

We begin by showing the critical Rayleigh number for onset of axisymmetric convection as a function of Reynolds number. For  $Re < 95$ , the bifurcation is to a steady state (i.e. the critical eigenvalue is real), and for  $Re > 95$ , the bifurcation is to an oscillatory state (i.e. the critical eigenvalue is complex). The thresholds in figure 2 are obtained by calculating the eigenvalues of the linearized problem for fixed values of  $(Re, Ra)$  and interpolating to find the value  $Ra_c(Re)$  at which the eigenvalue or its real part crosses zero. Thresholds were also obtained independently by fitting decay rates from the time-stepping code to exponentials and extrapolating these to zero.

Away from where the bifurcation changes its nature from steady to oscillatory,  $Ra_c$  should be a smooth function of  $Re$ . In fact,  $Ra_c$  must be an even function of  $Re$ , since reversing  $Re$  means reversing the direction of rotation of the upper and lower disks, an operation which leaves all dynamical properties unchanged, as was argued [25] in the context of Rayleigh-Bénard convection with throughflow. We fit  $Ra_c$  over the range of the steady bifurcation,  $0 \leq Re \leq 94$ , to an even polynomial, obtaining:

$$Ra_c \approx 2260 + (0.6243 \times Re)^2 + (0.0840 \times Re)^4 \quad (9)$$

We see that the von Kármán flow stabilizes the system against thermal convection. We contrast this with results from other mixed thermal/shear systems. In plane Poiseuille and Couette flow, the threshold for onset of longitudinal rolls is unaffected by the shear [23, 24]. (In our system, the rolls can be seen as longitudinal, since the axes of concentric rolls are azimuthal, as is the velocity of the bounding disks.) In Taylor-Couette flow (measured by the Taylor number  $Ta$ ) with a radial temperature gradient (measured by the Grashof number  $Gr$ ), the role of rotation and heating are reversed from our system. Any radial temperature gradient causes large-scale motion (a role played by  $Re$  in our system) and a bifurcation to Taylor vortices is seen at a finite rotation rate (a role played by  $Ra$  in our system). In experiments on this system [26],  $Ta_c$  decreases with  $Gr$ , in contrast to the increase in  $Ra_c$  with  $Re$  in our case.

The threshold value  $Ra_c(Re = 0) = 2260$  agrees well with those previously reported for the onset of convection for insulating sidewalls and aspect ratio  $\Gamma = 1$ , notably 2260 [1, 3], 2300 ( $\Delta Ra_c/Ra_c = 1.8\%$ ) [4], 2241 ( $\Delta Ra_c/Ra_c = 0.8\%$ ) [5], and 2250 ( $\Delta Ra_c/Ra_c = 0.4\%$ ) [6]. These authors all found the most unstable mode to be axisymmetric for cylindrical convection at this aspect

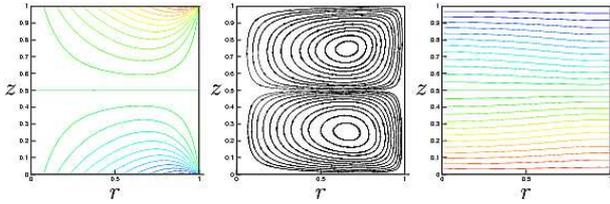


FIG. 3: (Color online) Basic flow at  $Re = 40$ ,  $Ra = 2000$ . From left to right:  $U_\theta$ ,  $\Psi$  and  $T$ . Ranges for  $U_r$  and  $U_z$  are  $[-0.05, 0.05]$  and  $[-0.04, 0.04]$ , respectively

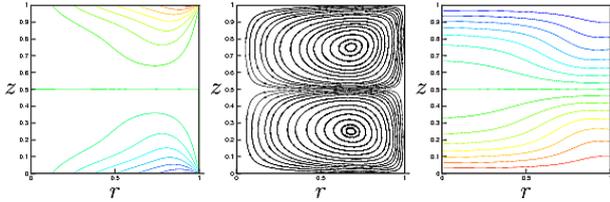


FIG. 4: (Color online) Basic flow at  $Re = 90$ ,  $Ra = 8000$ . From left to right:  $U_\theta$ ,  $\Psi$  and  $T$ . Ranges for  $U_r$  and  $U_z$  are  $[-0.11, 0.11]$  and  $[-0.07, 0.07]$ , respectively.

ratio.

Varying  $Re$  and investigating the thresholds of non-axisymmetric eigenmodes shows that the critical azimuthal wavenumber is  $m = 2$  for  $Re \geq 54.8$ . At  $Re = 0$ , the primary axisymmetric convective branch becomes unstable to an  $m = 2$  perturbation near  $Ra = 3000$  [4, 5]. For the pure von Kármán flow, i.e. with  $Ra$  set to zero and  $Re$  gradually increased from zero, the first instability occurs near  $Re = 300$  to an  $m = 2$  mode [15]. We nonetheless restrict our consideration here to the axisymmetric bifurcations and eigenmodes. There are a number of reasons for doing so. First, an understanding of the axisymmetric problem is important for completeness: branches which are unstable may nonetheless play a role in the dynamics. Secondly, the axisymmetric problem displays a number of fascinating phenomena, both linear and nonlinear, as we will see. Because of this, it has served as a test case for a numerical method using a reduced model [34]; a presentation of the results from fully resolved computations is a necessary complement to this study. Finally, additional physical effects, e.g. magnetic fields, can stabilize the axisymmetric configuration [40].

## V. STEADY CONVECTIVE STATES

We now discuss the nature of the steady flows below and above the convection threshold. The basic state, which is the analogue of the conductive state in the presence of von Kármán flow, is shown in figures 3 and 4 for  $Re = 40$ ,  $Ra = 2000$  and for  $Re = 90$ ,  $Ra = 8000$  respectively. The azimuthal velocity increases gradually from negative in the lower half to positive in the upper half of the cylinder, following the counter-rotating disks. It is

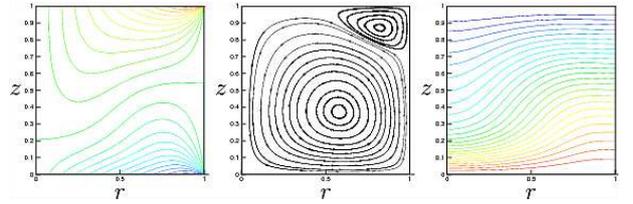


FIG. 5: (Color online) Convective state at  $Re = 40$ ,  $Ra = 4000$ . From left to right:  $U_\theta$ ,  $\Psi$  and  $T$ . Ranges for  $U_r$  and  $U_z$  are  $[-0.11, 0.19]$  and  $[-0.29, 0.11]$ , respectively.

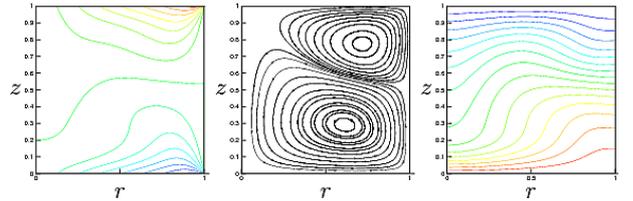


FIG. 6: (Color online) Convective state at  $Re = 90$ ,  $Ra = 12500$ . From left to right:  $U_\theta$ ,  $\Psi$  and  $T$ . Ranges for  $U_r$  and  $U_z$  are  $[-0.15, 0.18]$  and  $[-0.24, 0.10]$ , respectively.

singular at the corners  $r = 1$ ,  $z = 0, 1$  since the boundary conditions are discontinuous where the stationary cylinder meets the rotating disks. The large recirculating cells caused by Ekman pumping are prominent features in the  $(r, z)$  plane, carrying fluid radially outward along both rotating disks at  $z = 0, 1$ , then along the bounding cylinder, inwards along the midplane  $z = 1/2$ , and back towards the two disks along the axis. The azimuthal velocity and recirculating cells combine to yield fluid trajectories which are toroidal.

We may compare the basic flows in figures 3 and 4. For  $Re = 90$ , the temperature field is noticeably different from the linear conductive profile (8). The recirculating cells are stronger, with a maximal velocity of  $|U_r^{\max}| = 0.11$ , as compared to  $|U_r^{\max}| = 0.04$  for  $Re = 40$ . These stronger recirculating cells exert more influence on the temperature field. At  $r = 1$ , the isotherms are deviated towards the midplane as warm fluid converges upwards from the lower disk and cold fluid downwards from the upper disk. At  $r = 0$ , the isotherms are deviated again, here outwards towards the upper and lower disks. The azimuthal velocity gradients for  $Re = 90$  are more concentrated near the disks. The basic states shown in figures 3 and 4 are reflection-symmetric about the midplane: each is invariant under  $\kappa$ , as defined in (7).

The transition to convection breaks the reflection symmetry. In the absence of rotation, the transition occurs at  $Ra = 2260$ , as stated in section IV, and leads to a single roll which occupies the entire cavity. For  $Re = 40$ , transition occurs at  $Ra_c = 3125$  and the resulting convective state, shown in figure 5 for  $Ra = 4000$ , has one small roll (due to the weak von Kármán flow) in the upper right corner in addition to the main large convection roll. For  $Re = 90$ , the onset of convection occurs at  $Ra = 8622$

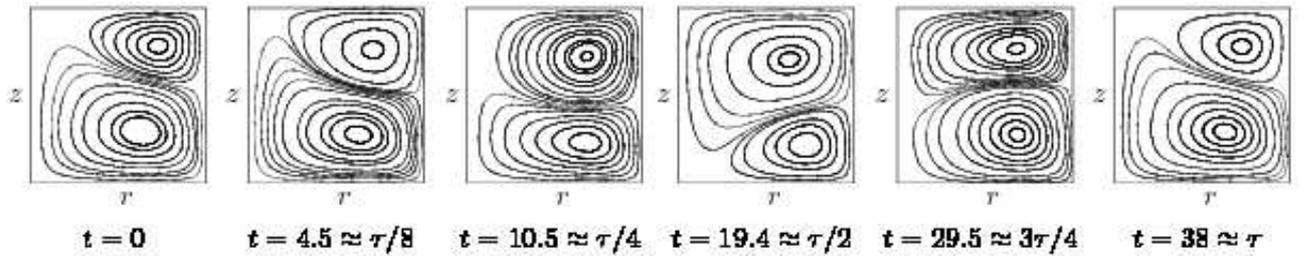


FIG. 7: Near-sinusoidal limit cycle at  $Ra = 20\,000$ ,  $Re = 110$ .  $\Psi$  is shown at times near  $0$ ,  $\tau/8$ ,  $\tau/4$ ,  $\tau/2$ ,  $3\tau/4$ ,  $\tau$  indicated on timeseries in figure 9. Times are given in units of  $1/\Omega$ . The overall limit cycle has reflection symmetry: the states in the second half of the cycle are related by reflection symmetry to those in the first half of the cycle. The vortex in the upper right corner is smaller during the first half of the cycle and larger during the second half. The streamfunction contours are not equally spaced, but instead chosen to illustrate topological features of the flow.

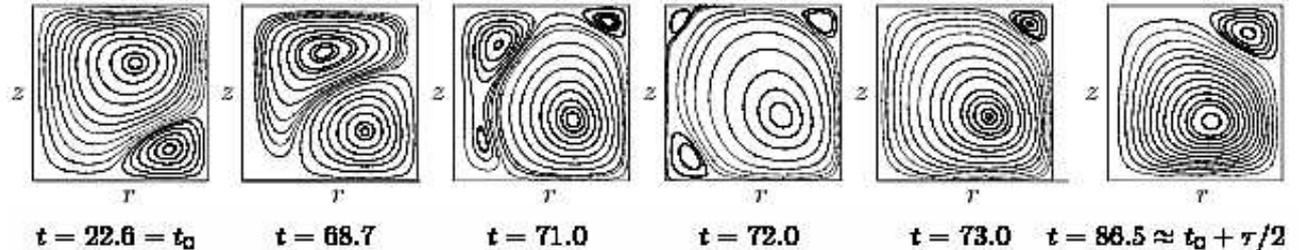


FIG. 8: Near-heteroclinic limit cycle at  $Ra = 20\,000$ ,  $Re = 63$ .  $\Psi$  is shown at times indicated on timeseries in figure 9:  $t = 22.6$ ; four instants in  $68 \leq t \leq 73$ , during which the flow changes a great deal; and  $t = 86.5$ , approximately a half-period  $\tau/2 \approx 57.5$  after  $t = 22.6$ . The vortex in the lower right corner grows and the vortex in the upper left corner folds and then divides into three small vortices. Two of these disappear, leaving a small vortex in the upper right corner. As in figure 7, the states in the second half of the cycle are related by reflection symmetry to those in the first half of the cycle and the streamfunction contours are chosen to illustrate topological features of the flow.

and the resulting convective state is shown in figure 6 at  $Ra = 12\,500$ . The influence of the counter-rotating disks is stronger than it is at  $Re = 40$  and so the recirculating rolls, resulting from the combined effects of Ekman pumping (which favors two equal rolls) and convection (which favors a single roll), are more equal in size. The consequences on  $U_\theta$  and  $T$  of these changes in  $\Psi$  can also clearly be seen in figures 5 and 6. (These in turn affect  $\Psi$  via the buoyancy and Ekman pumping effects, though to a lesser extent.)

Although the smaller cell in figures 5 and 6 is in the upper corner of the cylinder, the pitchfork bifurcation can produce a small cell at either the upper or lower corner. The breaking of reflection symmetry is also manifested by the azimuthal velocity: the  $U_\theta = 0$  contour intersects the axis near  $z = 0.2$  in both figures 5 and 6, in contrast to the basic flow, for which this contour is a straight line across the midplane at  $z = 0.5$ .

## VI. OSCILLATORY CONVECTIVE STATES

For  $Re > 95$ , the transition from the basic state is a Hopf bifurcation, creating oscillatory flows. We fix  $Ra = 20\,000$  and present two oscillatory states, or limit cycles, one at  $Re = 110$  in figure 7 and one at  $Re = 63$  in

figure 8. (Although for  $Re = 63$ , the first bifurcation as  $Ra$  is increased is to steady convection, there is another bifurcation at higher  $Ra$  to oscillatory convection and the resulting limit cycles are stable and connected to those created by the Hopf bifurcation for  $Re > 95$ , as we will see in section VII.) We call the limit cycles at  $Re = 110$  and  $Re = 63$  near-sinusoidal and near-heteroclinic, respectively, which we will also explain in section VII.

Figures 7 and 8 show meridional vortices jostling with one another. At  $Re=110$  (figure 7), the vortices are somewhat similar in size, with the vortex in the upper or lower outer corner alternatively becoming smaller and larger. At  $Re=63$  (figure 8), the principle vortex remains much larger throughout the limit cycle, with much smaller vortices on the periphery. At  $t = 71.0$ , there appear to be two vortices which both extend over the entire height, suggesting an interpretation of this limit cycle as a competition between one and two concentric radial vortices. This interpretation will be discussed further in section VIII.

Section V showed that the transition to steady convection breaks the reflection symmetry, creating two asymmetric states related by the reflection operator  $\kappa$ . The transition to oscillatory convection has an analogous property: each of the instantaneous states shown in figures 7 and 8 is asymmetric, and the second half of each

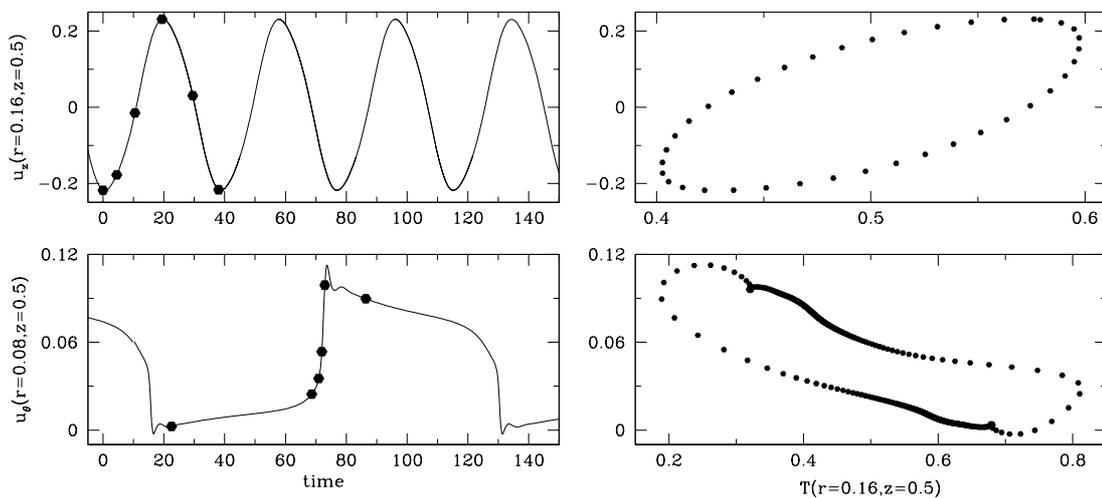


FIG. 9: Timeseries (left) and phase portraits (right) corresponding to limit cycles at  $Re = 20000$ . Dots in timeseries refer to visualizations in figures 7 and 8. Dots in phase portraits are plotted at equally spaced times so that their density reflects the rate at which the limit cycle is traversed. Top: at  $Re = 110$ , the oscillations are regular and near-sinusoidal. Bottom: at  $Re = 63$ , the oscillations have a much longer period and consist mainly of two long plateaus with abrupt gradients between them.

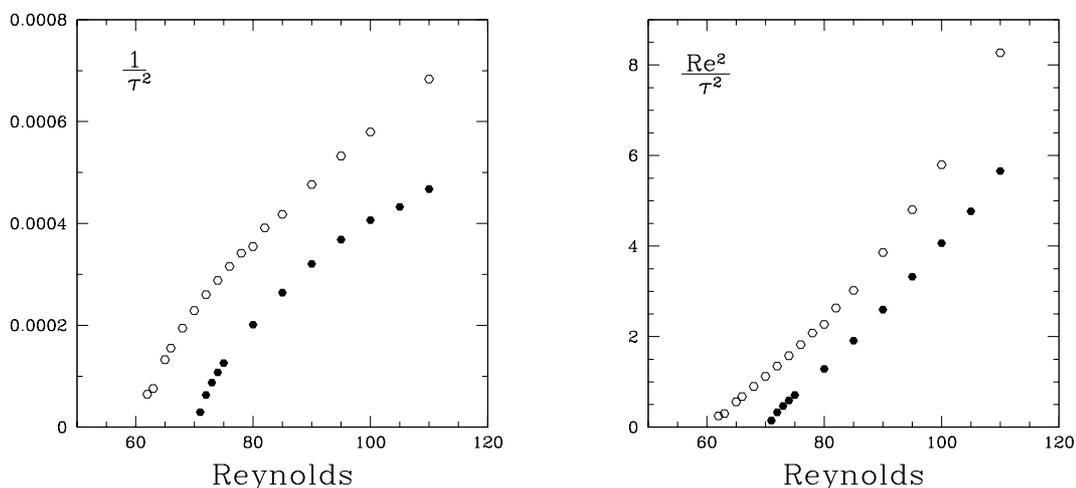


FIG. 10: Variation of square frequency with  $Re$ . Left:  $1/\tau^2$  for  $Ra = 20000$  (hollow dots) and for  $Ra = 18000$  (solid dots). Right:  $(Re/\tau)^2$  for  $Re = 20000$  (hollow dots) and for  $Ra = 18000$  (solid dots).

limit cycle is related to the first half by reflection, i.e.:

$$(U_r, U_\theta, U_z, T)(t + \tau/2) = \kappa(U_r, U_\theta, U_z, T)(t) \quad (10)$$

where  $\tau$  is the oscillation period.

To better understand the limit cycles in figures 7 and 8, we plot various scalar quantities –  $U_z(0.16, 0.5)$ ,  $U_\theta(0.08, 0.5)$  and  $T(0.16, 0.5)$  – as a function of time and of each other to create the timeseries and phase portraits of figure 9. Figure 9 clearly shows the very different character of the limit cycles at  $Re = 110$  and  $Re = 63$ . At  $Re = 110$ , the timeseries is fairly close to harmonic, and the phase portrait shows that the limit cycle is traversed at a fairly constant speed. In contrast, at  $Re = 63$ , the timeseries shows two phases during which change is very slow, punctuated by abrupt transitions. The phase portrait corroborates this: two regions of the limit cycle show

a dense accumulation of points, indicating a slow traversal of these regions.

We have seen that the period at  $Re = 63$  is about three times that at  $Re = 110$ ; it in fact approaches infinity as  $Re$  is decreased. (The approach by  $\tau$  to infinity implies that it is extremely sensitive to any change in the physical or numerical parameters. Thus, values of  $\tau$  are subject to a great deal of uncertainty.) This is demonstrated in figure 10, which plots the dependence of the square frequency (inverse square period) on  $Re$  for  $Ra = 20000$  and for  $Ra = 18000$ , for two different scalings: with the disk rotation period  $1/\Omega$  (left) and with the viscous diffusion time  $R^2/\nu$  (right).

## VII. BIFURCATION DIAGRAM

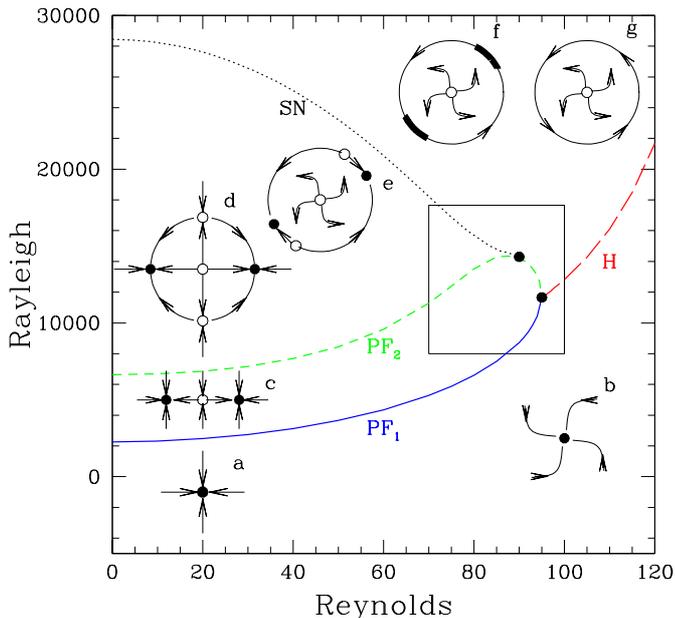


FIG. 11: (Color online) Curves of bifurcation points in the  $(Re, Ra)$  plane showing first pitchfork (blue, solid,  $PF_1$ ), second pitchfork (green, short-dashed,  $PF_2$ ), Hopf (red, long-dashed,  $H$ ), and saddle-node (black, dotted,  $SN$ ) bifurcations. Dots indicate codimension-two points, where curve  $PF_2$  meets curves  $SN$  (90,14300) and  $H$  (95,11750). An enlargement of the region inside the square is shown in figure 12. Phase portraits shown as insets. a, b) Below  $PF_1$  and  $H$ , the only solution is the stable basic flow, which is a node (a) or a spiral focus (b). c) Between  $PF_1$  and  $PF_2$ , the basic flow is unstable and there exist two stable asymmetric convective states. d, e) Between  $PF_2$  and  $SN$ , there exist five states: the unstable basic flow and two stable and two unstable convective states. f, g) Between  $SN$  and  $H$ , the unstable basic flow is surrounded by a stable limit cycle, which is near-heteroclinic near  $SN$  (f) and near-sinusoidal near  $H$  (g).

We have mentioned several features of the oscillatory states in section VI:

- The small vortex does not remain in the upper or lower corner, as it does for the steady convective states, but alternates between the two locations, as shown in figures 7 and 8.
- The character of the oscillations varies substantially, from near-sinusoidal to relaxational (plateaus punctuated by fast changes) as shown in figure 9.
- The period varies substantially with changes in  $Re$  or  $Ra$ , as shown in figure 10.

All of these features are typical of the bifurcation scenario we will now describe, and which is illustrated in figure 11.

For  $Re < 95$ , the threshold for convection, already shown in figure 2, corresponds to a pitchfork bifurcation

$PF_1$ . For  $Ra$  below  $PF_1$ , i.e. inset (a) in figure 11, the only solution is the basic flow, an example of which is given in figure 3. Proceeding clockwise, at  $PF_1$ , the basic state loses stability and gives rise to two symmetrically-related convective states, as illustrated in inset (c) and figure 5. A second pitchfork bifurcation,  $PF_2$ , occurs at a higher value of  $Ra$ . For  $Ra$  between curves  $PF_2$  and  $SN$ , in regions (d) and (e), there exist two additional steady states, both unstable and also related by the symmetry operation (7). The trajectories leaving the two unstable states terminate on the two stable states, forming an invariant set that resembles a circle or ellipse. Leaving  $PF_2$  and approaching  $SN$ , as shown in (e), the two stable and unstable states approach one other along this invariant set, finally merging and annihilating one another in two symmetrically related saddle-node bifurcations at  $SN$ .

Exactly at  $SN$ , trajectories spend an infinite period of time at the two locations at which the stable and unstable states merged, forming a limit cycle whose period is infinite: a heteroclinic cycle. For  $Ra$  above  $SN$ , but sufficiently nearby, as in inset (f), much of the time during the limit cycle is spent in the neighborhood of the former fixed points, as shown for  $Re = 63$ ,  $Ra = 20000$  in figure 8 and in the bottom half of figure 9. Leaving  $SN$ , by increasing  $Ra$  or  $Re$ , as in inset (g), the time spent in the vicinity of these points shortens, and the period decreases, as shown for  $Re = 110$ ,  $Ra = 20000$  in figure 7 and in the upper half of figure 9. Finally, for  $Re > 95$ , curve  $H$  corresponds to a Hopf bifurcation, below which the basic state is a stable spiral focus, as in inset (b).

Although the saddle-node bifurcation signals the merging of two pairs of steady states, it results in a limit cycle because of the connections between these steady states. (These connections in turn result from the formation of the steady states via two successive pitchfork bifurcations.) Under these circumstances, the saddle-node bifurcation is called a SNIPER, for Saddle-Node In a PERiodic orbit, or Saddle-Node Infinite PERiod; other names are a saddle-node homoclinic, a SNIC (Saddle-Node on Invariant Circle), or an Andronov bifurcation [41, 42]. It can be shown that, near a SNIPER bifurcation, the period  $\tau$  of the limit cycle varies like

$$\frac{1}{\tau^2} \sim \mu - \mu_c \quad (11)$$

where  $\mu - \mu_c$  can be either  $Ra - Ra_{SN}$  for fixed  $Re$  or  $Re - Re_{SN}$  for fixed  $Ra$  or any combination of the two. Although the square frequency is proportional to the distance from threshold sufficiently near the threshold for any scaling, figure 10 shows that, for our case, the linear dependence (11) holds over a wider range of  $Re$  when  $\tau$  is scaled by the viscous diffusion time (right) than when it is scaled by the disk rotation period (left).

Figure 11 shows two codimension-two points. At (95,11750), curves  $H$ ,  $PF_1$  and  $PF_2$  meet in a Takens-Bogdanov (TB) point. At (90,14300), curve  $SN$  terminates on  $PF_2$  at a hysteresis point, where the pitchfork bifurcation changes from supercritical to subcritical.

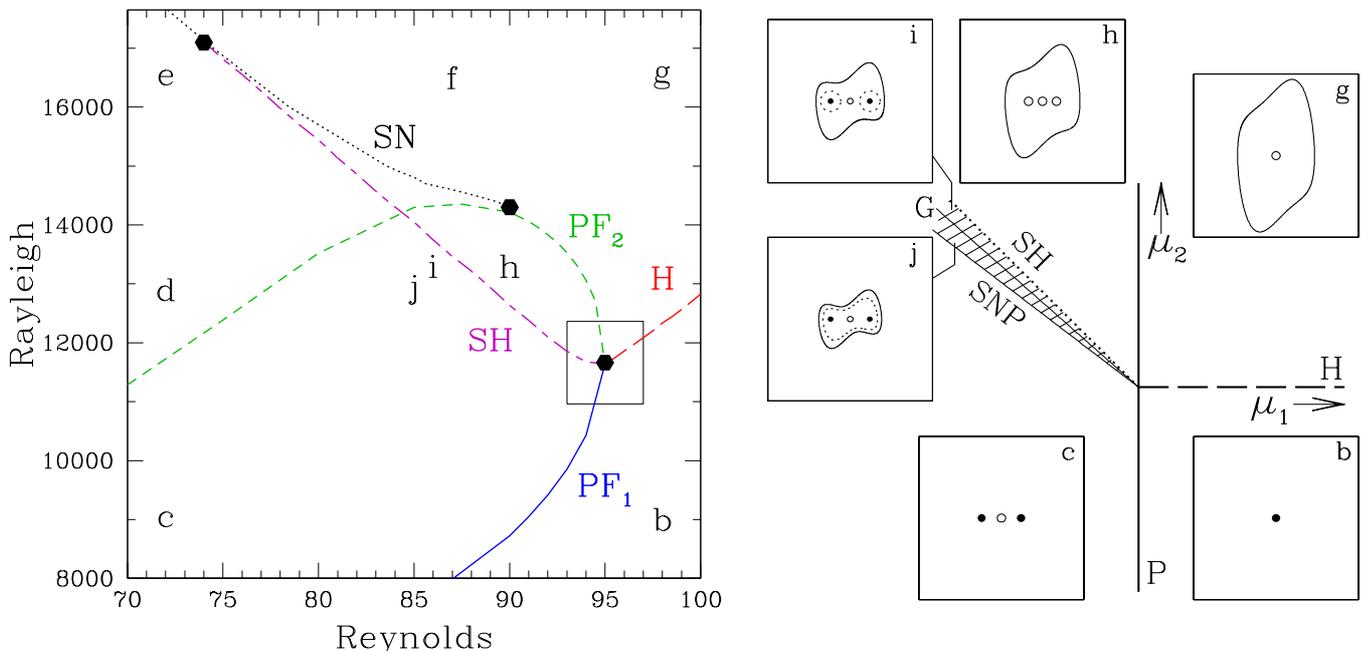


FIG. 12: (Color online) Left: enlargement of square in figure 11. The additional curve (violet, dash-dotted) indicates the secondary Hopf bifurcation  $SH$  which exists between codimension-two points (95, 11 750) (where it meets  $H$ ) and (74, 17 100) (where it meets  $SN$ ). Labels correspond to those of figure 11. Behavior in the square is described by the TB normal form (12). Right: behavior of the TB normal form (12). Shown are the pitchfork bifurcation  $P$ , Hopf bifurcation  $H$ , secondary Hopf bifurcation  $SH$ , as well as the saddle-node of periodic orbits  $SNP$  and the gluing bifurcation  $G$  which are contained in the unfolding of the TB point. Phase portraits b), c), g) as in figure 11. h) Between  $P$  and  $SH$  a stable limit cycle surrounds the unstable basic and convective flows. i) Between  $SH$  and  $G$ , small unstable limit cycles surround each stable convective state. j) Between  $G$  and  $SNP$ , two large limit cycles, one stable and one unstable, surround the three steady states. Hatched region indicates bistability between limit cycle and steady states.

(Because the TB point was calculating using eigenvalue computations rather than by specialized algorithms such as those in [38], we have been able to determine its location only to within about 0.5% in  $Re$  and 5% in  $Ra$ .) Between these two points, figure 11 shows curve  $PF_2$  separating region (c), in which the unstable basic state coexists with two stable steady states, and region (f,g) in which it coexists with a stable limit cycle. The pitchfork bifurcation  $PF_2$  cannot bring about such a transition. (The other paths between (c) and (f,g), either clockwise through  $PF_2$ , region d/e and  $SN$ ; or counter-clockwise through  $PF_1$ , region a/b and  $H$ , can bring about this transition.) This means that figure 11 is incomplete. The solution is to be found in the normal form of the Takens-Bogdanov codimension-two point in the presence of reflection symmetry [42, 43]:

$$\frac{dx}{dt} = y \quad (12a)$$

$$\frac{dy}{dt} = -\mu_1 x + \mu_2 y - \delta x^3 - x^2 y \quad (12b)$$

where  $\delta = \pm 1$ . We take  $\delta = +1$ , since it is this choice which reproduces the phenomena we see in the fluid-dynamical simulation.

The behavior of the normal form (12) is illustrated in

Steady state	Jacobian	Eigenvalues
(0, 0)	$\begin{pmatrix} 0 & 1 \\ -\mu_1 & \mu_2 \end{pmatrix}$	$\lambda_{\pm} = \frac{1}{2}\mu_2 \pm \sqrt{\frac{1}{4}\mu_2^2 - \mu_1}$
$(\pm\sqrt{-\mu_1}, 0)$	$\begin{pmatrix} 0 & 1 \\ 2\mu_1 & \mu_2\mu_1 \end{pmatrix}$	$\lambda_{\pm} = \frac{1}{2}(\mu_2 + \mu_1) \pm \sqrt{\frac{1}{4}(\mu_2 + \mu_1)^2 + 2\mu_1}$

TABLE I: Properties of the Takens-Bogdanov normal form.

figure 12 (right) and in table I. The  $TB$  codimension-two point is located at  $(\mu_1, \mu_2) = (0, 0)$  and the basic state at  $(x, y) = (0, 0)$ . The basic state undergoes a pitchfork bifurcation ( $\lambda_+ = 0$ ), analogous to our  $PF_1$  and  $PF_2$ , at  $\mu_1 = 0$  and a Hopf bifurcation ( $\mathcal{R}e(\lambda_{\pm}) = 0$ ), analogous to our  $H$  at  $\mu_2 = 0$  for  $\mu_1 > 0$ . Additionally, table I shows that the solutions  $(\pm\sqrt{-\mu_1}, 0)$  undergo a Hopf bifurcation at  $\mu_2 = -\mu_1$  for  $\mu_1 < 0$ . Indeed, we have calculated a secondary Hopf bifurcation  $SH$  for the hydrodynamic system, and indicate it in figure 12. This bifurcation is subcritical and the limit cycles it generates are unstable. Arnold [42, 43] has shown that system (12) contains two additional bifurcation curves emanating from the codimension-two point  $(\mu_1, \mu_2) = (0, 0)$ , one

a gluing bifurcation,  $G$ , and the other a saddle-node of periodic orbits,  $SNP$ . Although the curves  $SNP$  and  $G$  correspond to global bifurcations and cannot be calculated by examination of the eigenvalues in table I, they have been determined [42] to be  $\mu_2 = -0.752\mu_1$  for  $SNP$  and  $\mu_2 = -0.8\mu_1$  for  $G$ . (We have computed neither for our hydrodynamic system.) Together, bifurcations  $SNP$ ,  $G$ ,  $SH$  and  $P$  accomplish the transformation in the small square of figure 11, between the three steady states of region (c) to the stable limit cycle of region (g).

We describe the bifurcations and phase portraits in figure 12, proceeding around the TB point in the counter-clockwise direction. In region (b), the only solution is the stable basic state. The Hopf bifurcation  $H$  destabilizes the basic state and generates a stable limit cycle (g). The pitchfork bifurcation  $P$  is subcritical, and generates two unstable asymmetric states while reducing the number of unstable directions of the basic state from two to one (h). The secondary Hopf bifurcation  $SH$  is also subcritical, stabilizing the two asymmetric states while creating unstable limit cycles encircling each one (i). The gluing bifurcation  $G$  joins these two limit cycles to the symmetric state in a figure-eight which, afterwards, form a single unstable limit cycle encircling all three steady states (j). Curves  $SH$  and  $SNP$  delineate a narrow wedge in which there is bistability between the limit cycle and the steady convective states. The stable and unstable limit cycles annihilate in a saddle-node of periodic orbits  $SNP$ , leaving only the unstable basic flow and the stable asymmetric steady states (c). In crossing the pitchfork bifurcation  $P$ , we return to the single stable symmetric state of b). Figure 12 shows a third codimension-two point where  $SH$  and  $SN$  meet at (74,17100). Other codimension-two points which we have not located must mark the ends of curves  $G$  and  $SNP$  as well.

Although the analysis surrounding the TB point is important for completeness, we emphasize that, over most of the  $(Re, Ra)$  plane, it is the SNIPER bifurcation which marks the boundary between steady and oscillatory convection; other bifurcations come into play only between (95,11750) and (74,17100).

Figure 13 shows representative bifurcation diagrams. Branches are labelled by their stability index, i.e. the number of eigenvalues with positive real parts. The stability index changes by one at a pitchfork bifurcation (for the basic state) or a saddle-node bifurcation and by two for a Hopf bifurcation. We plot the temperature at a fixed point  $\bar{T} \equiv T(r = 1/2, z = 1/2)$  as a function of  $Ra$  for fixed  $Re$  along the bottom of the figure, and as a function of  $Re$  for fixed  $Ra$  along the right. To represent limit cycles, we plot  $[\frac{1}{\tau} \int_0^\tau \bar{T}(t)^2 dt]^{1/2}$ , the  $L_2$  norm of  $\bar{T}$  over an oscillation period  $\tau$ . The three codimension-two points at (74, 17100), (90, 14300) and (95,11750) and the limits of the pitchfork and saddle-node curves ( $Ra = 2260$  and  $28445$ ) delimit the ranges over which different types of bifurcation diagrams occur.

•  $Re = 60$  represents the large typical range  $0 \leq Re \leq 74$ : two successive supercritical pitchfork bifurcations gener-

ate two pairs of steady states which are annihilated by a saddle-node (SNIPER) bifurcation, leading to a limit cycle.

- $Re = 85$  represents [74, 90]: secondary Hopf bifurcations have been added to the diagrams.
- $Re = 93$  represents [90, 95]: the second pitchfork bifurcation has become subcritical.
- For  $Re > 95$  (not shown), there are no non-trivial steady states and a Hopf bifurcation leads directly to the limit cycle.
- $Ra = 5000$  represents the range  $2260 \leq Ra \leq 6640$ : a single pitchfork bifurcation leads to a pair of steady states as  $Re$  decreases.
- $Ra = 7000$  represents [6640, 11750]: an additional pitchfork bifurcation and pair of steady branches can be seen.
- $Ra = 13000$  represents [11750, 14300]: a primary and two secondary Hopf bifurcations have been added to the scenario.
- $Ra = 15000$  represents [14300, 17100]: the two pairs of non-trivial branches are no longer connected to the trivial branch via pitchfork bifurcations, but instead arise via saddle-node bifurcations.
- $Ra = 18000$  represents [17100, 28445]: the secondary Hopf bifurcations have disappeared.
- For  $Ra > 28445$  (not shown), there are no non-trivial steady states, but only a limit cycle generated by the Hopf bifurcation.

(The other bifurcations depicted in the enlargement of figure 12, which are present for the ranges  $74 \leq Re \leq 95$  and  $11750 \leq Ra \leq 17100$ , are omitted from figure 13 for clarity.)

Scenarios similar to those reported in this section have been computed in other hydrodynamic configurations. We review these studies and list the similarities and differences between these observations and the current results. In a cylinder with aspect ratio  $\Gamma = 5$ ,  $Pr = 10$  and thermally conducting sidewalls [7, 8], steady states are produced by two pitchfork bifurcations. When  $Ra$  is increased, a SNIPER bifurcation leads to a pattern of five concentric rolls traveling radially inwards. By reducing the sidewall conductivity, other saddle-node bifurcations on the same invariant circle appeared, creating stable steady four-roll states. No Hopf bifurcations or Takens-Bogdanov points were found. Such a bifurcation may also have been responsible for long-period oscillations observed in experimental timeseries, but for which the flow could not be visualized [2]. A SNIPER bifurcation is also found for  $\Gamma = 4$ ,  $Pr = 7$  in a study of rotating convection [30]. When rotation is turned on, the reflection symmetry (7) is broken and the pitchforks become imperfect, but the SNIPER bifurcation survives, with a limit cycle created by one saddle-node bifurcation instead of two simultaneous ones.

Siggers [9] carried out an extensive survey over  $4 \leq \Gamma \leq 10$ ,  $Pr = 0.1$  with (less realistic but more tractable) stress-free boundary conditions. Like [34], the numerical method used an expansion in eigenfunctions of the

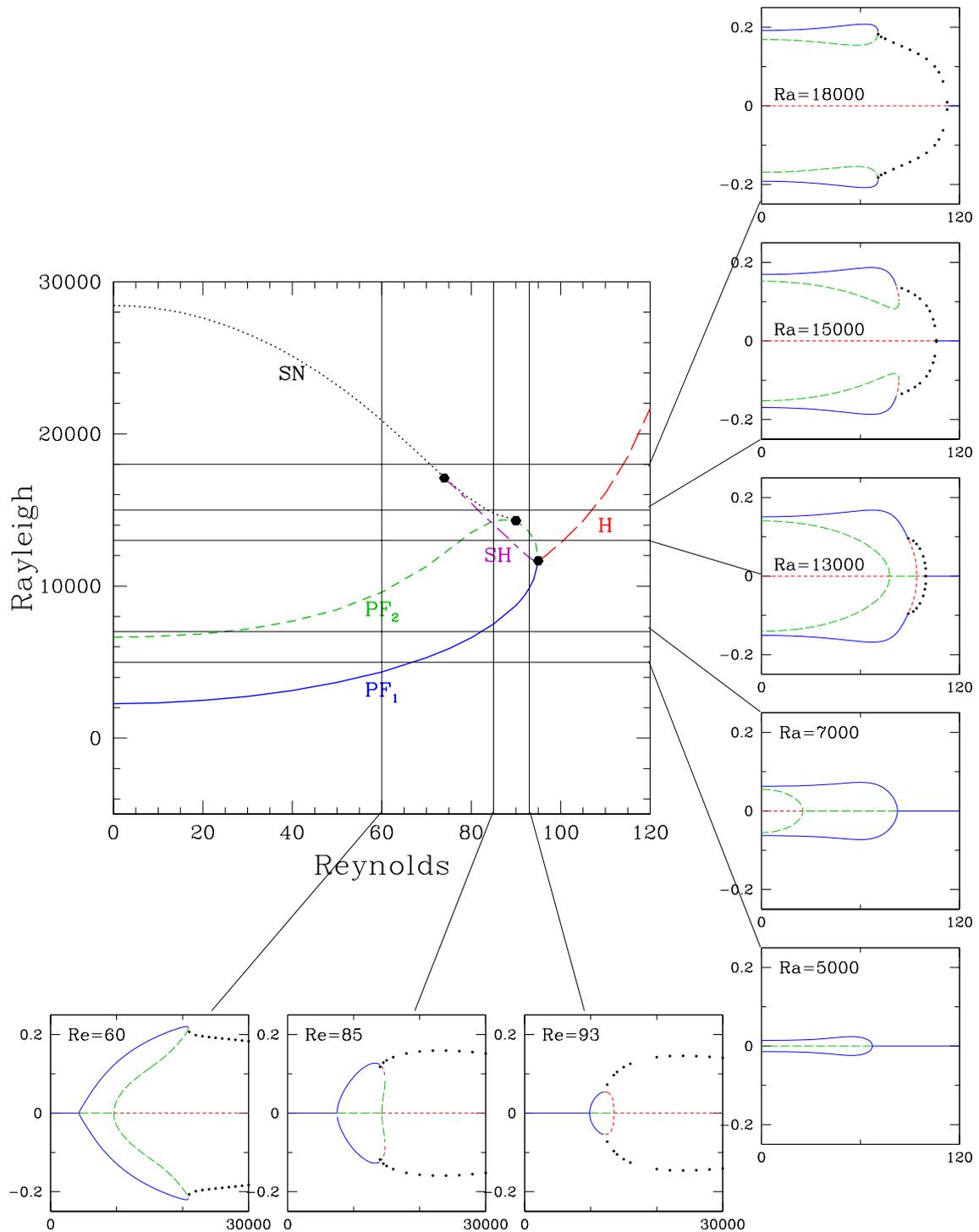


FIG. 13: (Color online) Bifurcation diagrams along lines in central  $(Re, Ra)$  diagram.  $T(1/2, 1/2)$  is plotted as a function of  $Ra$  for fixed  $Re$  in the diagrams along the bottom, and as a function of  $Re$  for fixed  $Ra$  in the diagrams in the column on the right. Diagrams contain steady branches with zero (blue, solid), one (green, long-dashed) and two (red, short-dashed) unstable directions. Black dots indicate limit cycles.

linearized problem. In [9], 20–30 eigenfunctions were retained, a number sufficiently small to allow automatic tracking of bifurcations. A phase diagram very similar in its large-scale features to that described in this section was found: limit cycles compete with steady states that are created by two pitchfork bifurcations and destroyed by saddle-node bifurcations. However, unlike in our case, the basic state never undergoes a Hopf bifurcation like our  $H$ ; the only Hopf bifurcations are those, like our  $SH$ , which create small limit cycles, like those in inset (i) of figure 12, sometimes called *vaccillation*. The large limit cycle is created by neither a SNIPER nor a Hopf bifurcation, but rather by a saddle-node of periodic orbits (as in inset (j) of our figure 12) or by a heteroclinic bifurcation.

SNIPER bifurcations are also observed in situations with more complicated time dependence, in which the saddle-node bifurcations annihilate pairs of limit cycles to initiate flow along a torus. This is the case in a study of Rayleigh-Bénard convection with modulated rotation [31]. The basic state is a large-scale axisymmetric and reflection-symmetric time-periodic flow. As  $Ra$  or the modulation amplitude of the rotation is increased, pitchfork bifurcations produce pulsating target patterns which give way to radially travelling target patterns via a SNIPER bifurcation. Abshagen et al. [44, 45] have carried out experimental and computational studies of secondary bifurcations in Taylor-Couette flow in a small-aspect-ratio cylinder. The transitions they consider are between nonaxisymmetric rotating and modulated rotating waves. However, if one of the time scales is filtered out by taking a Poincaré map and the reflection symmetry is redefined to include rotation by  $\pi$  around the cylinder axis, their results can be seen as entirely analogous to ours. Varying the aspect ratio and the Reynolds number, the region in which modulated rotating waves exist is bounded by curves of SNIPER and Hopf bifurcations, except over a small region in which homoclinic bifurcations and small-amplitude modulations mediate the transition.

### VIII. EIGENVALUES AND EIGENVECTORS

We now focus on the behavior of the eigenvalues and eigenvectors of the basic flow as the Rayleigh and Reynolds numbers are varied. Although the linear stability analysis of a problem is usually presented before its bifurcation scenario, we will focus here on some features of the eigenvalues and eigenvectors that are independent of the bifurcation scenario.

To obtain the linear stability equations, we substitute

$$(\Psi, V_\theta, T)(r, z) + e^{\sigma t}(\psi, v_\theta, \Theta)(r, z) \quad (13)$$

into the governing equations (5)-(6), where  $(\Psi, V_\theta, T)$  is a steady solution of (5)-(6), and  $(\psi, v_\theta, \Theta)$  is a perturbation with growth rate  $\sigma$ . We note that the temperature perturbation  $\Theta$  is unrelated to the azimuthal

angle  $\theta$  while the meridional velocity perturbation is  $v_r \mathbf{e}_r + v_z \mathbf{e}_z = \mathbf{e}_\theta / r \times \nabla \psi$ . Retaining only the linear terms leads to:

$$\begin{aligned} \sigma D^2 \psi &= -(v_r \partial_- + v_z \partial_z) D^2 \Psi \\ &\quad - (V_r \partial_- + V_z \partial_z) D^2 \psi + \left( \nabla^2 - \frac{1}{r^2} \right) D^2 \psi \\ &\quad - \frac{Ra}{Pr} \partial_r \Theta + 2 \partial_z \left( \frac{V_\theta v_\theta}{r} \right) \end{aligned} \quad (14a)$$

$$\begin{aligned} \sigma \Theta &= -(v_r \partial_r + v_z \partial_z) T \\ &\quad - (V_r \partial_r + V_z \partial_z) \Theta + \frac{1}{Pr} \nabla^2 \Theta \end{aligned} \quad (14b)$$

$$\begin{aligned} \sigma v_\theta &= -(v_r \partial_+ + v_z \partial_z) V_\theta - (V_r \partial_+ + V_z \partial_z) v_\theta \\ &\quad + \left( \nabla^2 - \frac{1}{r^2} \right) v_\theta \end{aligned} \quad (14c)$$

with homogeneous boundary conditions:

$$\begin{aligned} \psi = \partial_z \psi = 0 &\iff v_r = v_z = 0, \\ v_\theta = 0, \Theta = 0 &\text{ at } z = 1 \end{aligned} \quad (15a)$$

$$\begin{aligned} \psi = \partial_z \psi = 0 &\iff v_r = v_z = 0, \\ v_\theta = 0, \Theta = 0 &\text{ at } z = 0 \end{aligned} \quad (15b)$$

$$\begin{aligned} \psi = \partial_r \psi = 0 &\iff v_r = v_z = 0, \\ v_\theta = 0, \partial_r \Theta = 0 &\text{ at } r = 1 \end{aligned} \quad (15c)$$

$$\begin{aligned} \psi = D^2 \psi = 0 &\iff v_r = \partial_r v_z = 0, \\ v_\theta = 0, \partial_r \Theta = 0 &\text{ at } r = 0 \end{aligned} \quad (15d)$$

The Reynolds number appears via the inhomogeneous boundary conditions on the nonlinear equations (6a,b), making  $V_\theta$  proportional to  $Re$ . In order to make this dependence on  $Re$  explicit, we will scale  $V_\theta$  (but not  $V_{r,z}$  or  $\Psi$ ) by  $Re$ :

$$V_\theta = Re U_\theta \quad (16)$$

This scaling can be used even when  $Re = 0$  since  $V_\theta$  is proportional to  $Re$ . We then rewrite (14) in a more compact matrix form as follows:

$$\begin{aligned} \sigma \begin{pmatrix} D^2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \psi \\ \Theta \\ v_\theta \end{pmatrix} &= \mathcal{L} \begin{pmatrix} \psi \\ \Theta \\ v_\theta \end{pmatrix} \\ &= \left( \begin{array}{cc|c} \mathcal{L}_{\psi\psi} & \mathcal{L}_{\psi\Theta} & Re \mathcal{L}_{\psi v_\theta} \\ \mathcal{L}_{\Theta\psi} & \mathcal{L}_{\Theta\Theta} & 0 \\ \hline Re \mathcal{L}_{v_\theta\psi} & 0 & \mathcal{L}_{v_\theta v_\theta} \end{array} \right) \begin{pmatrix} \psi \\ \Theta \\ v_\theta \end{pmatrix} \end{aligned} \quad (17)$$

The operators  $\mathcal{L}_{\psi\psi}, \dots$  which comprise  $\mathcal{L}$ , and which depend on  $Re$  implicitly (and weakly) via the steady state, are listed below. The boxes surround terms which are non-zero when  $Re = 0$  and the steady state is the con-

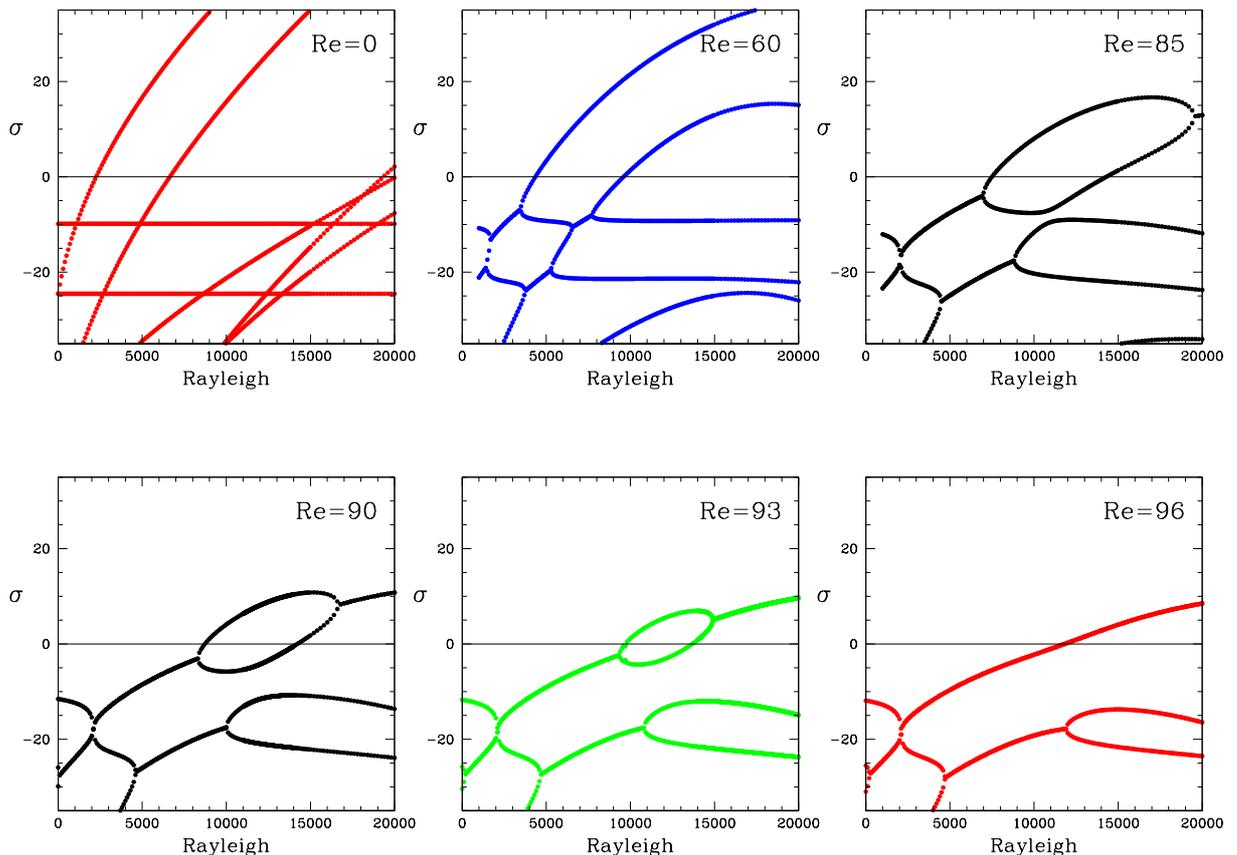


FIG. 14: (Color online) Real part  $\sigma$  of leading eigenvalues as a function of  $Ra$  for six values of  $Re$ . For  $Re = 0$ , the eigenvalues are real and cross transversely. Zero crossings at  $Ra = 2260$  and  $Ra = 6640$  correspond to pitchfork bifurcations. Purely thermal and azimuthal eigenvalues at  $\sigma = -9.87$  and  $\sigma = -24.6$ , respectively, are independent of  $Ra$ . When  $Re > 0$ , the transverse crossings become complex conjugate pairs, over  $Ra$  intervals which widen with increasing  $Re$ . By  $Re = 96$ , the bifurcating eigenvalue is a complex conjugate pair, leading to a Hopf bifurcation at  $Ra = 11\,856$ .

ductive solution  $\mathbf{V} = 0$ ,  $T = 1 - z$ .

$$\begin{aligned} \mathcal{L}_{\psi\psi}\psi &= -\frac{1}{r}((\partial_z\psi)\partial_- - (\partial_r\psi)\partial_z)D^2\Psi \\ &\quad - (V_r\partial_- + V_z\partial_z)D^2\psi + \left(\nabla^2 - \frac{1}{r^2}\right)D^2\psi \end{aligned} \quad (18a)$$

$$\mathcal{L}_{\psi\Theta}\Theta = \frac{Ra}{Pr}\partial_r\Theta \quad (18b)$$

$$\mathcal{L}_{\psi v_\theta}v_\theta = \frac{2}{r}\partial_z(U_\theta v_\theta) \quad (18c)$$

$$\mathcal{L}_{\Theta\psi}\psi = -\frac{1}{r}(\partial_z\psi)\partial_r T + \frac{1}{r}(\partial_r\psi)\partial_z T \quad (18d)$$

$$\mathcal{L}_{v_\theta\psi}\psi = -\frac{1}{r}((\partial_z\psi)\partial_+ - (\partial_r\psi)\partial_z)U_\theta \quad (18e)$$

$$\mathcal{L}_{\Theta\Theta}\Theta = -(V_r\partial_r + V_z\partial_z)\Theta + \frac{1}{Pr}\nabla^2\Theta \quad (18f)$$

$$\mathcal{L}_{v_\theta v_\theta}v_\theta = -(V_r\partial_+ + V_z\partial_z)v_\theta + \left(\nabla^2 - \frac{1}{r^2}\right)v_\theta \quad (18g)$$

Figure 14 shows the real part of the leading eigenvalues (in units of the inverse viscous diffusive time) as a function of  $Ra$  for several fixed values of  $Re$ , while figure 15 shows some of the corresponding leading eigenvectors for  $Re = 0$  and for  $Re = 96$ , both for  $Ra = 10\,000$ . Figure 14 shows several eigenvalues which are independent of  $Ra$  when  $Re = 0$ . The origin of these is well known and easily understood. For  $Re = 0$ , equations (17)-(18) show that the linear stability problem becomes:

$$\begin{aligned} \sigma \begin{pmatrix} D^2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \psi \\ \Theta \\ v_\theta \end{pmatrix} \\ = \begin{pmatrix} (\nabla^2 - \frac{1}{r^2})D^2 & -\frac{Ra}{Pr}\partial_r & 0 \\ -\frac{1}{r}\partial_r & \frac{1}{Pr}\nabla^2 & 0 \\ 0 & 0 & (\nabla^2 - \frac{1}{r^2}) \end{pmatrix} \begin{pmatrix} \psi \\ \Theta \\ v_\theta \end{pmatrix} \end{aligned} \quad (19)$$

with the homogeneous boundary conditions (15). Purely

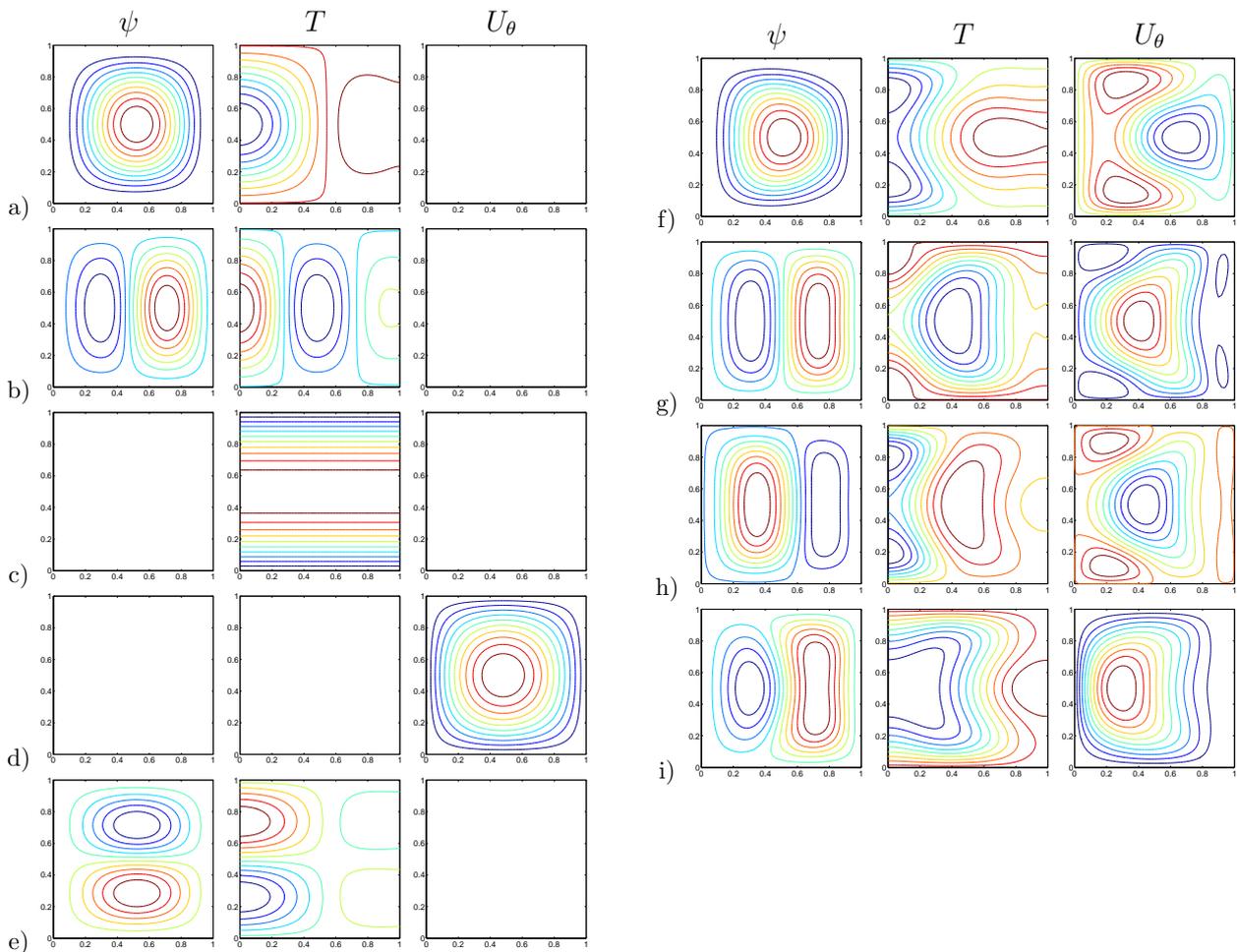


FIG. 15: (Color online) Leading eigenvectors at  $Ra = 10000$ . Left column:  $Re = 0$ . a,b) eigenvectors responsible for transition to convection with one and two toroidal rolls, respectively, with  $\sigma = 38.5$  and  $\sigma = 15.9$ . c,d) thermal and azimuthal velocity eigenvectors, respectively, with  $\sigma = -9.87 \approx -\pi^2$  and  $\sigma = -24.6 \approx -\pi^2 - j_{11}^2$ . e) eigenvector with two vertically stacked rolls, with  $\sigma = -21.2$ . Right column:  $Re = 96$ . f,g) first complex conjugate pair, with  $\sigma \pm i\omega = -2.24 \pm i3.45$ . h,i) second complex conjugate pair, with  $\sigma \pm i\omega = -19.2 \pm i4.35$ .

thermal eigenvectors have  $\mathbf{v} = 0$ , leading to:

$$\left. \begin{aligned} 0 &= -\frac{Ra}{Pr} \partial_r \Theta \\ \sigma \Theta &= \frac{1}{Pr} \nabla^2 \Theta \\ 0 &= \Theta|_{z=0,1} = \partial_r \Theta|_{r=0,1} \end{aligned} \right\} \Rightarrow \Theta = \sin(k\pi z) \quad \text{with } \sigma = -(k\pi)^2 \quad (20)$$

The first of these thermal eigenvalues is seen in figure 14 ( $Re = 0$ ) as the horizontal line very close to its analytic value of  $-\pi^2 = -9.8696$ . The corresponding thermal eigenvector is shown in figure 15c. Another set of solutions contains only azimuthal velocity: with  $\Theta = \psi = 0$  and  $v_\theta$  a solution to

$$\left. \begin{aligned} \sigma v_\theta &= (\nabla^2 - \frac{1}{r^2}) v_\theta \\ 0 &= v_\theta|_{z=0,1} = v_\theta|_{r=0,1} \end{aligned} \right\} \Rightarrow v_\theta = \sin(k\pi z) J_1(rj_{1n}) \quad \text{with } \sigma = -((k\pi)^2 + j_{1n}^2) \quad (21)$$

where  $J_1$  is the first Bessel function and  $j_{1n} = 3.8317, 7.0156, \dots$  is one of the zeros of  $J_1$ . The first of these

azimuthal velocity eigenvalues is the horizontal line seen in figure 14 ( $Re = 0$ ), again very close to its analytically computed value of  $-(\pi^2 + 3.8317^2) = -24.551$ . The corresponding azimuthal velocity eigenvector is shown in figure 15d.

The other eigenvalues shown in figure 14 ( $Re = 0$ ) increase with  $Ra$ . The zero crossings of the two largest eigenvalues are associated with the pitchfork bifurcations discussed extensively in the previous section; these take place at  $Ra = 2260$  and  $Ra = 6640$ . Their associated eigenvectors are shown in figure 15a,b, for  $Ra = 10000$ , where their eigenvalues are 38.5 and 15.9, and they are seen to contain one and two concentric radial rolls, respectively. The leading eigenvectors for higher  $Re$  are similar. The convective states in figures 5 and 6 do not resemble figure 15a, even though they result from a pitchfork bifurcation involving this eigenvector. This is because the nonlinear steady states in figures 5 and 6

are a superposition of these eigenvectors and the basic flows shown in figures 3 and 4. The eigenvector in figure 15e, with two vertically stacked rolls, is associated with eigenvalue  $\sigma = -21.2$ , between that of the thermal and azimuthal eigenvalues at  $Re = 10\,000$ .

An overall feature of the eigenvalues for  $Re = 0$  that can be seen in figure 14 is that they are all real and cross transversely. Indeed, it is well-known that for Rayleigh-Bénard convection (i.e.  $Re = 0$ ), all of the eigenvalues are real. When  $Re > 0$ , the eigenvalue crossings which occur for  $Re = 0$  become complex conjugate pairs. (Because the real parts of the eigenvalues are shown in figure 14, complex conjugate pairs appear as a fusion of two eigenvalue curves.) As  $Re$  increases, the  $Ra$  intervals over which the eigenvalues are complex widen. By  $Re = 96$ , the real eigenvalues responsible for the pitchfork bifurcations have merged into a complex conjugate pair whose real part crosses zero at  $Ra = 11\,856$  at the Hopf bifurcation point. At  $Re = 95$  (not shown), the fusion of two real eigenvalues into a complex conjugate pair and the zero-crossing occur simultaneously at  $Ra = 11\,750$ : this is the Takens-Bogdanov codimension-two point.

Figure 15f, g shows the real and imaginary part of the complex conjugate pair which is the leading eigenvector for  $Re = 96$ . The fact that this complex pair at  $Re = 96$  originates in the fusion of the leading real eigenvectors at lower  $Re$  is made strikingly clear when the streamfunctions  $\psi$  in figure 15f, g are compared with those of figure 15a, b. Note that the choice of the real and imaginary parts of an eigenvector is arbitrary, since an eigenvector can be multiplied by any complex number. The particular choice here is imposed by the normalization  $\text{Im}(\psi(r = 1/2, z = 1/2)) = 0$ . The decomposition of the complex eigenvector  $\psi$  into two vectors containing one and two concentric radial rolls shows that the limit cycle will involve competing radial structures. As with the steady states, the lack of resemblance between figures 15f, g and the limit cycle in figure 7 is due to the fact that these are superpositions of the basic state and the eigenvectors.

We investigate the progression of the eigenvalues from real to complex as  $Re$  increases by examining the matrix in (17), which is block diagonal for  $Re = 0$ , as shown in (19). Its eigenvalues and eigenvectors thus consist of two sets: those of the thermal convection problem (upper left submatrix) and those of the azimuthal problem (lower right submatrix). The convective, thermal and azimuthal eigenvalues cross transversely because (19) has no off-diagonal terms and because the eigenvalues within each of these sub-problems, each associated with a different spatial structure, do not cross one another.

This behavior resembles that seen near the Takens-Bogdanov point which occurs in binary fluid or thermosolutal convection. In the binary/thermosolutal case, the pair of eigenvectors which interact have different origins: one can be viewed as arising primarily from thermal convection and the other from solutal convection [46]. In the Rayleigh-Bénard/von Kármán case studied here, the

eigenvectors which become complex both arise from thermal convection; the difference between them is their spatial structure, shown in figure 15a,b.

In the binary/thermosolutal case, the transverse crossings undergo two different fates, depending on the sign of the separation parameter  $S$ , which describes whether the thermal and solutal convection act in concert or in opposition. For positive  $S$ , the eigenvalue curves separate into two hyperbolas, in what is called avoided crossings, and remain real. For negative  $S$ , the eigenvalues join in complex conjugate pairs, as in figure 14. These two cases can be understood in terms of a  $2 \times 2$  matrix, whose off-diagonal terms are of the same sign if  $S$  is positive and of opposite signs if  $S$  is negative. Here, we have not attempted such an analysis, but we can conclude that the off-diagonal matrices

$$\begin{pmatrix} Re \mathcal{L}_{\psi v_\theta} \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Re \mathcal{L}_{v_\theta \psi} & 0 \end{pmatrix} \quad (22)$$

are in some sense of opposite signs, since the coupling they cause between the convective and azimuthal velocity eigenvectors leads to complex eigenvalues. The coupling between the convective and the purely thermal eigenvectors must also be of this type. In [46], a calculation of the sign of the coupling terms is presented for the binary/thermosolutal case, involving projecting onto the eigenvectors of the diagonal submatrices.

## IX. DISCUSSION

We have shown that a transition analogous to the onset of convection occurs in an axisymmetric cylindrical container subjected to vertical gradients in temperature and azimuthal velocity, i.e. in Rayleigh-Bénard/von Kármán flow. Differential rotation, which causes mixing via Ekman pumping, delays the onset of convection. The transition is a pitchfork bifurcation, leading to steady convection, for  $Re < 95$  and a Hopf bifurcation, leading to oscillatory convection, for  $Re > 95$ . Between these two types of convection, over most of the  $(0 \leq Re \leq 120, 0 \leq Ra \leq 30\,000)$  parameter space, the transition occurs via a SNIPER bifurcation, in which the stable steady states meet a pair of unstable steady states and mutually annihilate, leaving a limit cycle in their wake. Over a small portion of the parameter space, the scenario is more complicated and involves several global bifurcations. (Section IV mentions some consequences of relaxing the imposition of axisymmetry.)

The linear stability analysis of the axisymmetric Rayleigh-Bénard/von Kármán problem also shows interesting features. We have traced the way in which differential rotation couples the eigenvalue branches, which are real for Rayleigh-Bénard convection, in such a way that they become complex. The close resemblance between the leading real pair of eigenmodes for  $Re < 95$  and the leading complex pair for  $Re > 95$  supports the idea that

a common basis of eigenvectors could be used to make the reduced model studied in the companion paper [34] more economical.

Since its discovery by Andronov and Leontovich [41], the SNIPER bifurcation has appeared in a number of ODE systems used to model chemical reactions, notably in excitable media and the Belousov-Zhabotinsky reaction [47], and population biology, for example predator-prey systems. In the hydrodynamic and PDE context, the SNIPER bifurcation was observed in simulations of axisymmetric Rayleigh-Bénard convection [7, 8, 30, 31]. Convective states with different radial wavelengths compete, much as occurs in the present Rayleigh-Bénard/von Kármán configuration. The closest analogy with the bifurcation scenario we observe is found in the small-aspect-ratio study of Taylor-Couette flow by [44, 45], in which rotating and modulated rotating waves play the role of our steady states and limit cycles, respectively.

Recently, the SNIPER bifurcation has been the subject of renewed attention as a possible explanation for the reversals of the earth's magnetic field. A dynamo engendered by bulk fluid motion was first produced in a laboratory experiment of the von Kármán flow in sodium (VKS), an electrically conducting fluid [19]. This VKS experiment shows reversals of the polarity of the magnetic field which bear some similarity to that of the ter-

restrial field. An explanation involving a SNIPER bifurcation in a low-dimensional dynamical system with noise has been put forward to explain these reversals [20, 21]. This manifestation of the SNIPER bifurcation also provides some possible justifications for studying the axisymmetric flow. In the VKS experiment, the mean of the highly turbulent flow is axisymmetric, even though the instantaneous flow is not. A different mechanism for magnetic field reversals has been proposed [48], in which noise is added to a low-dimensional dynamical system displaying a Takens-Bogdanov point. The Takens-Bogdanov point, separating a steady from a Hopf bifurcation point, also plays an essential role in our system, as it does in the investigations of axisymmetric Rayleigh-Bénard convection by [9] and of Taylor-Couette flow by [44, 45]. It seems plausible that the TB point and the hysteresis point terminating the SNIPER bifurcation curve form part of the unfolding of a codimension-three point.

Our study was first intended to determine the effect of the von Kármán flow on the onset of Rayleigh-Bénard convection. Over the course of the investigation, we encountered a bifurcation scenario leading to oscillations, which should and does occur quite generally, whenever consecutive pitchfork bifurcations take place. It should therefore be of interest to the dynamical-systems community as well as to fluid dynamicists.

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