

Klein-Gordon Equation with Coulomb Potential in the Presence of a Minimal Length

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Abstract

We study the Klein-Gordon equation for Coulomb potential $V(r) = (-Ze^2)/r$ in quantum mechanics with a minimal length. The zero energy solution is obtained analytically in momentum space in terms of Heun's functions. The asymptotic behavior of the solution shows that the presence of a minimal length regularize the potential in the strong attractive regime, $Z > 68$. The equation with nonzero energy is established in a particular case in the first order of the deformation parameter; it is a generalized Heun's equation.

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I. INTRODUCTION

In recent years, a lot of attention has been attracted to the study of physical problems within the formalism of quantum mechanics with a generalized uncertainty relation [1], including a minimal length, see, for instance Refs. [2], and for a recent detailed review and a large list of references in connection with this subject, see, Ref. [3]. This modified version of quantum mechanics is based on the following deformed commutation relations between position and momentum operators [4, 5]:

$$\begin{aligned}
 [\widehat{X}_i, \widehat{P}_j] &= i\hbar[(1 + \beta\widehat{P}^2)\delta_{ij} + \beta'\widehat{P}_i\widehat{P}_j], & (\beta, \beta') > 0, \\
 [\widehat{P}_i, \widehat{P}_j] &= 0, \quad [\widehat{X}_i, \widehat{X}_j] = i\hbar\frac{2\beta - \beta' + \beta(2\beta + \beta')\widehat{P}^2}{1 + \beta\widehat{P}^2} (\widehat{P}_i\widehat{X}_j - \widehat{P}_j\widehat{X}_i). & (1)
 \end{aligned}$$

These commutators lead to the, so-called, generalized uncertainty principle (GUP), which implies the existence of a nonzero minimal uncertainty in position (minimal length) given, in N dimensions, by [4]

$$(\Delta X_i)_{\min} = \hbar\sqrt{(N\beta + \beta')}, \quad \forall i. \quad (2)$$

The idea of modifying the Heisenberg uncertainty relation in such a way that it incorporate a minimal length has been, first, proposed in quantum gravity and string theory [6–8], where the minimal length is supposed to be on the order of the Planck scale. However, it was argued that in nonrelativistic or relativistic quantum mechanics [4, 9], the minimal length may be viewed as an intrinsic scale characterizing the system under study. Furthermore, it was shown in Ref. [9] that this elementary length regularize in a natural way the strong attractive inverse square potential, known to be singular in quantum mechanics.

In this work, we study the Klein-Gordon (KG) equation for Coulomb potential, $V(r) = (-Ze^2)/r$, in the presence of a minimal length. In ordinary KG equation, this problem becomes singular when $Z > 68$, and the potential must be regularized by introducing a cutoff and modifying the interaction at short distances [10]. It is thus interesting to examine to what extent the introduction of a minimal length in the formalism regularizes the strong attractive Coulomb potential.

II. ORDINARY KG EQUATION FOR COULOMB POTENTIAL: MOMENTUM SPACE TREATMENT

In QM with a minimal length, momentum space is more convenient than coordinate space [1]. So, for the sake of further discussion, we give here the solution to ordinary KG equation for Coulomb potential in the momentum representation. We shall be interested in the singularity structure of this equation to illustrate how the strong attractive potential becomes singular. The KG equation for the potential $V(r) = (-Ze^2)/r$ reads

$$\left(E\widehat{R}^2 + Ze^2E\widehat{R} + Ze^2 \right) \psi(p) = \widehat{R}^2 (m^2c^4 + p^2c^2) \psi(p). \quad (3)$$

In the case of zero angular momentum quantum number ($\ell = 0$), the distance squared and distance operators act in momentum space as

$$R^2\psi(p) = -\hbar^2 \left(\frac{d^2}{dp^2} + \frac{2}{p} \frac{d}{dp} \right) \psi(p), \quad R\psi(p) = i\hbar \left(\frac{d}{dp} + \frac{1}{p} \right) \psi(p).$$

Replacing in Eq. (3), we get

$$\begin{aligned} \hbar^2 (\epsilon^2 + c^2p^2) \frac{d^2\psi(p)}{dp^2} + \left(\frac{2}{p}\hbar^2\epsilon + 2i\hbar ZEe^2 + 6\hbar^2c^2p \right) \frac{d\psi(p)}{dp} + \\ + \left(Z^2e^4 + \frac{2i\hbar ZEe^2}{p} + 6\hbar^2c^2 \right) \psi(p) = 0. \end{aligned} \quad (4)$$

where $\epsilon^2 = m^2c^4 - E^2$.

The two solutions of this equation behave at infinity as

$$\psi_1(p) \sim p^{-\frac{5}{2}-\mu}, \quad \psi_2(p) \sim p^{-\frac{5}{2}+\mu}, \quad (5)$$

where $\mu = \sqrt{\frac{1}{4} - \frac{e^4Z^2}{\hbar^2c^2}}$.

If $Z < 68$, the first solution falls off more rapidly than the other, and thus it is selected as the physical solution. It can also be shown that the average value of the operator \widehat{R}^2 is divergent at infinity ($\langle \widehat{R}^2 \rangle_{\psi_2} \rightarrow \infty$) [11]. When $Z < 68$, the general solution is a linear combination of the two solutions that behave in the same manner. Accordingly, the wave function will depend on an arbitrary phase parameter, which is a common feature of singular potentials [13].

The complete solution to Eq. (4) is written in terms of the hypergeometric function as

$$\psi(p) = Ap^{-1} (1 + (ic/\epsilon)p)^{-\frac{3}{2}-\mu} F\left(\frac{3}{2} + \mu, \frac{1}{2} - w + \mu, 2\mu + 1; 2/(1 + (ic/\epsilon)p)\right), \quad (6)$$

where F is a hypergeometric function, A is a normalization constant and $w = \frac{ZE\epsilon^2}{\hbar c}$.

The energy spectrum can be obtained by setting

$$1/2 - w + \mu = -n, \quad n = 0, 1, 2, \dots, \quad (7)$$

to ensure that the wave function (6) be square integrable. In this case, the hypergeometric series reduces to a polynomial.

Equation (7) constitutes the quantization condition of the energy, which give the well-known discrete energy spectrum of Coulomb potential [10].

In the case $Z > 68$, the parameter μ becomes imaginary and thus the spectral condition (7) fails. In order to obtain a discrete spectrum, the potential must be regularized by introducing a cutoff at short distances [10]. Another alternative approach was discussed in Ref. [14] to deal with the strong attractive Coulomb potential in the context of Dirac equation to overcome the dependence of the problem on the number Z .

In the following section we will show that, when a minimal length is introduced in the K-G equation, there will be any difference between the strong and weak attractive regimes.

III. KG EQUATION FOR COULOMB POTENTIAL WITH A MINIMAL LENGTH

In the literature, one of the most used representations of the position and momentum operators satisfying Eqs. (1) is [5, 9]

$$\hat{X}_i = i\hbar \left[(1 + \beta p^2) \frac{\partial}{\partial p_i} + \beta' p_i p_j \frac{\partial}{\partial p_j} + \gamma p_i \right], \quad \hat{P}_i = p_i, \quad (8)$$

where γ is a small positive parameter related to β , β' ; it does not affect the observable quantities.

In the case $\ell = 0$ and $\gamma = 0$, representation (8) leads to the following expression of the distance squared operator [5, 9]:

$$\hat{R}^2 = \sum_{i=1}^3 \hat{X}_i \hat{X}_i = (i\hbar)^2 \left\{ [1 + (\beta + \beta') p^2]^2 \frac{d^2}{dp^2} + \frac{2}{p} [1 + (\beta + \beta') p^2] [1 + (2\beta + \beta') p^2] \frac{d}{dp} \right\}. \quad (9)$$

As it was shown in Sec. 2, the singularity of the strong attractive Coulomb potential manifests at infinity in momentum space (or equivalently, at short distances in coordinate space). In this limit, the solution of the K-G equation does not depend on the energy E . It

is thus interesting to begin our study by examining the effect of the minimal length on the zero-energy K-G equation.

A. The zero energy case

Inserting Eq. (9) in the KG equation (3), we obtain after some calculation the following equation:

$$\begin{aligned} & \frac{d^2\psi(p)}{dp^2} + \frac{2}{p} \left\{ 4 \left[\frac{p^2 + \frac{1}{2}m^2c^2}{p^2 + m^2c^2} \right] - \frac{1 + \beta'p^2}{1 + (\beta + \beta')p^2} \frac{d\psi(p)}{dp} \right\} \\ & + \left\{ \frac{6 + (10\beta + 6\beta')p^2}{[1 + (\beta + \beta')p^2]} + \frac{Z^2e^4/\hbar^2c^2}{[1 + (\beta + \beta')p^2]^2} \right\} \frac{\psi(p)}{p^2 + m^2c^2} = 0. \end{aligned} \quad (10)$$

At infinity, the two solutions of this equation behave as

$$\psi_1(p)_{p \rightarrow \infty} \sim p^{-3-2\beta/(\beta+\beta')}, \quad \psi_2(p)_{p \rightarrow \infty} \sim p^{-2}. \quad (11)$$

These behaviors are completely different from that of the non deformed case, see Eq. (5). Both solutions are independent of the atomic number Z ; moreover, the second solution ψ_2 does not depend on the deformation parameters and falls off more slowly than ψ_1 . It follows that ψ_1 is manifestly the physical solution regardless the value of Z . Consequently, in the presence of a minimal length, there is no difference between the strong ($Z > 68$) and the weak ($Z < 68$) regimes of the potential. This might be interpreted as a signal of regularization of the potential.

Let us now end this section by mentioning that Eq. (10) can be transformed to the following Heun's differential equation, by using the change of variable $\xi = \frac{(\beta+\beta')p^2}{1+(\beta+\beta')p^2}$, and the transformation $\psi(\xi) = (1 - \xi)f(\xi)$:

$$\frac{d^2f(\xi)}{d\xi^2} + \left(\frac{c}{\xi} + \frac{e}{\xi - 1} + \frac{d}{\xi - \xi_0} \right) \frac{df(\xi)}{d\xi} + \left(\frac{ab\xi + q}{\xi(\xi - 1)(\xi - \xi_0)} \right) f(\xi) = 0, \quad (12)$$

with the parameters

$$\begin{aligned} a &= \frac{1}{2}(3 - \omega_1 - \nu), \quad b = \frac{1}{2}(3 - \omega_1 + \nu), \quad c = \frac{3}{2}, \quad d = 2, \quad e = \frac{1}{2} - \omega_1, \\ \omega_1 &= \frac{2\beta}{\beta + \beta'}, \quad \nu = \left[(\omega_1 - 1)^2 - \frac{4k}{1 - 2\omega_2} \right]^{\frac{1}{2}}, \quad q = - \left(\frac{3}{2} + \frac{k}{1 - 2\omega_2} \right), \\ \xi_0 &= \frac{2\omega_2}{2\omega_2 - 1}, \quad k = \frac{Z^2e^4}{4\hbar^2c^2}, \quad \omega_2 = \frac{1}{2}(\beta + \beta')m^2c^2 \end{aligned} \quad (13)$$

The solution to Eq. (12), which is regular at $\xi = 0$ is [15]:

$$\psi(\xi) = A(1 - \xi) H(\xi_0, q, a, b, c, d, \xi). \quad (14)$$

It is important to mention that solution (14) reduces to an hypergeometric function in the particular case $\beta = \beta'$ as follow:

$$\psi_{\beta=\beta'}(\xi) = A(1 - \xi) F(a, b, c; \xi/\xi_0), \quad (15)$$

with the parameters

$$a = 5/4 - \nu/2, \quad b = 5/4 + \nu/2, \quad c = 3/2, \quad \nu = \sqrt{1/4 - 4k/(1 - 2\omega_2)}.$$

B. The nonzero energy case

The KG equation (3) can not be established in the presence of a minimal length, because the definition of the operator \widehat{R} is not obvious as long as the \widehat{R}^2 operator, given by Eq. (9), is not factorizable in general. However, it was shown in Ref. [12] that, in the particular case $\beta' = 2\beta$, this operator can be factorized in the first order of the deformation parameter, and thus the KG equation can be obtained in this special case. In fact, by using [12]

$$\widehat{R}^2 = (i\hbar)^2 \left\{ (1 + 6\beta p^2) \frac{d^2}{dp^2} + \frac{2}{p} (1 + 7\beta p^2) \frac{d}{dp} \right\} + O(\beta^2), \quad (16)$$

$$\widehat{R} = i\hbar \left[(1 + 3\beta p^2) \frac{d}{dp} + \frac{1}{p} (1 + \beta p^2) \right] + O(\beta^2), \quad (17)$$

equation (3) takes the form

$$(p^2 + \epsilon^2)(1 + 6\beta p^2) \frac{d^2 \varphi(p)}{dp^2} + \{2\beta p(p^2 + \epsilon^2) + 4p(1 + 6\beta p^2) + 2i\omega(1 + 3\beta p^2)\} \frac{d\varphi(p)}{dp} \\ \{-2\beta(p^2 + \epsilon^2) - 2(1 + 6\beta p^2) + 4(1 + 7\beta p^2) - 4i\omega\beta p + k\} \varphi(p) = 0, \quad (18)$$

where

$$\psi(p) = \frac{1}{p} \varphi(p), \quad \omega = \frac{Ze^2 E}{\hbar c^2}, \quad \epsilon^2 = m^2 c^2 - \frac{E^2}{c^2}, \quad k = \left(\frac{Ze^2}{\hbar c}\right)^2.$$

By making the change of variable,

$$x = \frac{1}{2} \left(1 - i\sqrt{6\beta p} \right),$$

equation (18) can be transformed to the following generalized Heun's equation [16]:

$$\frac{d^2 \varphi}{dx^2} + \left(\frac{c}{x} + \frac{d}{(x-1)} + \frac{e}{(x-x_1)} + \frac{f}{(x-x_2)} \right) \frac{d\varphi}{dx} + \left(\frac{abx^2 + \rho_1 x + \rho_2}{x(x-1)(x-x_1)(x-x_2)} \right) \varphi = 0, \quad (19)$$

With the Fuchsian condition,

$$a + b + 1 = c + d + e + f.$$

The parameters of Eq. (19) are given by

$$\begin{aligned} a &= 1, \quad b = \frac{7}{3}, \quad \rho_1 = \left(-\frac{7}{3} - \frac{\omega\sqrt{6\beta}}{3} \right), \quad \rho_2 = \frac{\beta\epsilon^2}{2} + \frac{\omega\sqrt{6\beta}}{6} + \frac{1}{12} - \frac{k}{4}, \\ c &= \left(\frac{1}{6} + \frac{\omega\sqrt{6\beta}}{2(1-6\beta\epsilon^2)} \right), \quad d = \frac{1}{6} - \frac{\omega\sqrt{6\beta}}{2(1-6\beta\epsilon^2)}, \quad e = \left(2 + \frac{\omega(1-3\beta\epsilon^2)}{(1-6\beta\epsilon^2)\epsilon} \right), \\ f &= \left(2 - \frac{\omega(1-3\beta\epsilon^2)}{(1-6\beta\epsilon^2)\epsilon} \right), \quad x_1 = \left(\frac{1}{2} + \frac{\epsilon\sqrt{6\beta}}{2} \right), \quad x_2 = \frac{1}{2} - \frac{\epsilon\sqrt{6\beta}}{2}. \end{aligned}$$

Equation (19) belongs to the class of Fuchsian equations: it is a linear homogeneous second-order differential equation with five singularities $z = 0, 1, x_1, x_2, \infty$, all regular. So, it admits power series solutions in the neighborhood of each singular point. However, to the best of our knowledge, the analytic solutions to the generalized Heun's equation (19) are not known in the literature, i.e., the recurrence relation that determines the coefficients of the series was not established for equations of type (19). It follows that the formulation of a physical problem with this kind of equation is interesting in itself. This might motivate profound studies on such Fuchsian equations with five singularities.

To end this section, let us discuss the effect of the minimal length in the singularity structure of Eq. (19). It can be checked that, in the limit $p \gg 1$, the two linearly independent solutions are

$$\psi_1 \sim p^{-10/3}, \quad \psi_2 \sim p^{-2}.$$

This result confirms what we have obtained in the zero energy case; the two solutions do not have the same behavior as in the ordinary case. Consequently we can always reject ψ_2 even for the strong attractive regime $Z > 68$, so that the wave function will not contain an arbitrary phase, characterizing singular potentials [13]. Remains to mention that the difference between the physical solution ψ_1 and the zero energy solution (5) is due to the fact that Eq. (18) is obtained in the first order in β , however the zero energy equation (10) is exact.

IV. SUMMARY

We have studied the KG equation for Coulomb potential in quantum mechanics with a minimal length. The zero energy solution is obtained analytically in momentum space;

it is a Heun's function, which reduces to a hypergeometric function in the case $\beta = \beta'$. The asymptotic behavior of the solution at large momenta showed that the presence of a minimal length regularize the potential in the strong attractive regime. The nonzero energy equation was established in the particular case $\beta' = 2\beta$ in the first order of the deformation parameter. It is transformed to a canonical form of Fuchsian equations, namely, a generalized Heun equation. The behavior of the solution confirms the regularizing effect of the minimal length.

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