

Singular inverse-square potential: renormalization and self-adjoint extensions for medium to weak coupling

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(Date textdate; Received textdate; Revised textdate; Accepted textdate; Published textdate)

Abstract

We study the radial Schrödinger equation for a particle of mass m in the field of the inverse-square potential α/r^2 in the medium-weak-coupling region, i.e., with $-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4$. By using the renormalization method of Beane *et al.*, with two regularization potentials, a spherical square well and a spherical δ shell, we illustrate that the procedure of renormalization is independent of the choice of the regularization counterterm. We show that, in the aforementioned range of the coupling constant α , there exists at most one bound state, in complete agreement with the method of self-adjoint extensions. We explicitly show that this bound state is due to the attractive square-well and delta-function counterterms present in the renormalization scheme. Our result for $2m\alpha/\hbar^2 = -1/4$ is in contradiction with some results in the literature.

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I. INTRODUCTION

Once considered devoid of physical interest [1], the nonrelativistic attractive singular inverse-square interaction was recently studied in connection with many problems of great physical interest such as Efimov states [2], dipole-bound anions in polar molecules [3], capture of matter by black holes [4], atoms interacting with a charged wire [5], and the dynamics of a dipole in a cosmic string background [6]. From a more formal viewpoint, the singular inverse-square interaction provides a simple example of the renormalization-group limit cycle in nonrelativistic quantum mechanics [7–10]. It has been furthermore shown [8] that the method of self-adjoint extensions [11, 12] for solving the Schrödinger equation for a particle of mass m with a strong attractive inverse-square potential $V(r) = \alpha/r^2$ ($2m\alpha/\hbar^2 < -1/4$) is equivalent to the renormalization method (R-method) of Beane *et al.* [7]. However, the method of self-adjoint extensions also applies to the “medium-weak”-coupling range ($-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4$) and leads to the possible existence of a single bound state even for a repulsive ($\alpha > 0$) potential [12, 13]. It has been argued [14] that this counterintuitive result is an undesirable feature of this method; the authors of Ref. [14] discuss an alternative regularization which removes this bound state. This problem is of physical importance in the study of bound anions, where no bound states have been experimentally observed for values of the strength of the dipole field below some critical value (for a review, see Ref. [15]) and may be relevant to the study of the near horizon structure of black holes [4]. For the sake of completeness, let us mention that the singular α/r^2 potential was studied in the framework of quantum mechanics with a minimal length uncertainty relation [16]. In this framework, the potential is regularized in a natural way and the bound states only exist in the strong-coupling regime.

The purpose of this paper is to apply the R-method of Ref. [7] to study the singular inverse square potential in the medium-weak-coupling range. Following Ref. [9], we show, using square-well and delta-function counterterms that, as in the strong-coupling regime, the R-method is equivalent to the self-adjoint extensions technique. The R-method clearly shows that a bound state may arise because the strength of the regularization potential is such that, for a given choice of the self-adjoint extension, it leads to a bound state even in the absence of long range inverse-square interaction ($\alpha = 0$) or in the presence of a repulsive ($0 < 2m\alpha/\hbar^2 \leq 3/4$) long-range interaction. The R-method thus elucidates the origin of this

“obscure” [12] bound state, which can, however, always be suppressed by a suitable choice of the self-adjoint extension. In the special case $2m\alpha/\hbar^2 = -1/4$, which may be relevant to the near horizon structure of black holes [4], the R-method predicts the existence of at most one bound state, in agreement with Ref. [13] but in sharp contrast with Refs. [4] where it was claimed that there are infinitely many bound states.

The rest of this paper is organized as follows. In Sec. II, we consider the square-well regularization potential and study the renormalization of the α/r^2 interaction in the medium-weak-coupling range. In Sec. III we renormalize the α/r^2 potential using the spherical δ shell regularization potential. We summarize our results in a brief concluding section.

II. SPHERICAL SQUARE-WELL REGULARIZATION

The starting point of our study is the s -wave reduced radial Schrödinger equation for one particle of mass m in the external potential $V(r)$,

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dr^2} + V(r)\right) u(r) = Eu(r), \quad (1)$$

where $V(r)$ is given by

$$\begin{aligned} V(r) &= \frac{\alpha}{r^2}, & (r > R), \\ &= -\lambda \frac{\hbar^2}{2mR^2}, & (r < R). \end{aligned} \quad (2)$$

The dimensionless “coupling constant” λ is a positive function of the short-distance cutoff R and the long-range coupling constant α is taken such as $(-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4)$.

While the renormalization of the strong attractive α/r^2 potential ($2m\alpha/\hbar^2 < -1/4$) was studied extensively in the literature (see, e.g., Refs. [7–10]), the weak-medium-coupling regime ($-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4$) is studied here.

Following Refs. [7–9], we first consider Eq. (1) in the zero energy case ($E = 0$). The solution that is continuous at $r = R$ is given by

$$\begin{aligned} u_0(r) &= A \left[(r/r_0)^{1/2+\nu} - c (r/r_0)^{1/2-\nu} \right], & r > R \\ &= \frac{A}{\sin \sqrt{\lambda}} \left[(R/r_0)^{1/2+\nu} - c (R/r_0)^{1/2-\nu} \right] \sin(k_0 r), & r < R, \end{aligned} \quad (3)$$

where $k_0^2 = \lambda/R^2$, $\nu = (2m\alpha/\hbar^2 + 1/4)^{1/2}$, A is a normalization constant, c is an arbitrary constant, and r_0 is an arbitrary scale. The usual matching condition of the derivative $u'(r)$ at $r = R$ yields

$$\sqrt{\lambda} \cot \sqrt{\lambda} = \frac{(\frac{1}{2} + \nu) (R/r_0)^{2\nu} - c (\frac{1}{2} - \nu)}{(R/r_0)^{2\nu} - c}. \quad (4)$$

Equation (4) gives the values of the short-range coupling constant λ as a function of the cutoff R .

Now we turn to the bound-state spectrum ($E = -\frac{\hbar^2}{2m}k^2$). The solution to Eqs (1) and (2) is then given by

$$\begin{aligned} u(r) &= Nr^{1/2}K_\nu(kr), & r > R, \\ &= N' \sin(Kr), & r < R, \end{aligned} \quad (5)$$

where N and N' are normalization constants, $K_\nu(z)$ is a modified Bessel function and

$$K^2 = k_0^2 - k^2. \quad (6)$$

Matching logarithmic derivatives at $r = R$ now gives

$$KR \cot(KR) = \frac{1}{2} + kR \frac{K'_\nu(kR)}{K_\nu(kR)}. \quad (7)$$

In the low-energy limit ($kR \ll 1$), one has $KR \simeq k_0R = \sqrt{\lambda}$. Using the formula [17]

$$\begin{aligned} K_\nu(z) &= \frac{\pi}{2} \frac{I_{-\nu}(z) - I_\nu(z)}{\sin \nu\pi}, \\ I_\nu(z) &\sim \left(\frac{1}{2}z\right)^\nu / \Gamma(\nu + 1), \quad z \rightarrow 0, \end{aligned} \quad (8)$$

Eq. (7) takes the following form in the limit ($kR \ll 1$):

$$\sqrt{\lambda} \cot \sqrt{\lambda} = \frac{1}{2} + \nu \frac{\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{kR}{2}\right)^{2\nu} + 1}{\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{kR}{2}\right)^{2\nu} - 1}. \quad (9)$$

By identifying Eqs. (4) and (9) we then get

$$k = \frac{2}{r_0} \left[\frac{\Gamma(1+\nu)}{c\Gamma(1-\nu)} \right]^{1/2\nu}, \quad (10)$$

with binding energy $E = -\hbar^2 k^2 / 2m$. Note that the condition $kR \ll 1$ implies that formula (10) only holds for $R/r_0 \ll 1$.

Formula (10) is in complete agreement with the result of Ref. [12] using the method of self-adjoint extensions. As one can easily see, in the range $(-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4)$ of the coupling constant α , there exists a single bound state even if the inverse-square interaction is zero ($\nu = 1/2$) or repulsive ($1/2 < \nu \leq 1$). The origin of this bound state lies in the short-range modification of the interaction (this will be clearer in the next section). On the other hand, formula (10) shows that this bound state can always be eliminated by choosing the arbitrary constant $c < 0$ and thus is not a compulsory feature of the self-adjoint extension method.

Let us now examine the case $\nu = 0$ (the coupling constant α takes the critical value $-\hbar^2/8m$), which requires special consideration. Let us recall that the inverse-square potential with this particular value of α would have some physical importance as it appeared in the study of the near horizon structure of black holes [4].

Denoting $v_0(r)$ the zero energy solution for $\nu = 0$, we now have

$$\begin{aligned} v_0(r) &= A (r/r_0)^{1/2} [1 + c \ln (r/r_0)], \quad r > R, \\ &= \frac{A}{\sin \sqrt{\lambda}} (R/r_0)^{1/2} [1 + c \ln (R/r_0)] \sin(k_0 r), \quad r < R. \end{aligned} \quad (11)$$

The matching condition of the derivative $u'(r)$ at $r = R$ now yields

$$\sqrt{\lambda} \cot \sqrt{\lambda} = \frac{1}{2} + \frac{c}{1 + c \ln (R/r_0)}. \quad (12)$$

The bound state wave function $v(r)$ is now

$$\begin{aligned} v(r) &= N r^{1/2} K_0(kr), \quad r > R, \\ &= N' \sin(Kr), \quad r < R, \end{aligned} \quad (13)$$

where N and N' are normalization constants, $k^2 = -\frac{2m}{\hbar^2} E$ and $K^2 = k_0^2 - k^2$ with $k_0^2 = \lambda/R^2$.

As in the case $\nu \neq 0$, the continuity of logarithmic derivatives at $r = R$ now gives the condition

$$KR \cot(KR) = \frac{1}{2} + kR \frac{K'_0(kR)}{K_0(kR)}. \quad (14)$$

For very shallow bound states, $kR \ll 1$ ($KR \simeq k_0 R = \sqrt{\lambda}$), we use the formula [17]

$$K_0(z) \sim -\ln(z), \quad z \rightarrow 0, \quad (15)$$

and Eq. (14) is written in the following form:

$$\sqrt{\lambda} \cot \sqrt{\lambda} = \frac{1}{2} + \frac{1}{\ln(kR)}. \quad (16)$$

From Eqs. (12) and (16), we get

$$k = \frac{1}{r_0} \exp(1/c), \quad (17)$$

with $R/r_0 \ll 1$. We obtain a single bound state with binding energy given by

$$E = -\frac{\hbar^2}{2mr_0^2} \exp(2/c). \quad (18)$$

This result is in agreement with Ref. [13] but is in strong disagreement with some studies of the structure of near horizon black holes [4] advocating the existence of infinitely many bound states for $\nu = 0$.

III. REGULARIZATION WITH A δ -SHELL POTENTIAL

As it was noted in Ref. [9], renormalization theory should lead to identical results for low-energy observables independently of the regularization potential. It has been furthermore shown that the spherical δ -shell potential is particularly convenient in the regularization of the strong attractive inverse-square potential and gives the same results as in the square-well regularization. In this section, we consider the medium-weak-coupling range of the potential in the R-method with a spherical δ shell potential. The conclusions of Sec. II will all be confirmed.

The potential that we consider now is given by

$$\begin{aligned} V(r) &= \frac{\alpha}{r^2} & (r > R), \\ &= -\frac{\lambda\hbar^2}{2mR} \delta(r - R^-) & (r < R). \end{aligned} \quad (19)$$

with $(-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4)$. The long-range attractive α/r^2 potential in Eq. (19) is cut off at a short distance R by a spherical shell with radius $r = R^-$ infinitesimally close to but smaller than R [9].

As in the previous section, we first solve the zero energy Schrödinger equation in order to establish the renormalization-group flow. The solution to Eq. (1) with the potential (19) that is continuous at $r = R$ in the case $E = 0$ is given by

$$\begin{aligned}
u_0(r) &= A \left[(r/r_0)^{1/2+\nu} - c (r/r_0)^{1/2-\nu} \right], & r > R, \\
&= A \left[(R/r_0)^{-1/2+\nu} - c (R/r_0)^{-1/2-\nu} \right] (r/r_0), & r < R,
\end{aligned} \tag{20}$$

where $\nu = (2m\alpha/\hbar^2 + 1/4)^{1/2}$, A is a normalization constant, c is an arbitrary constant and r_0 is an arbitrary scale.

The usual boundary condition from the delta-function potential at $r = R$ for the derivative of the wave function is then [9]

$$\lim_{r \rightarrow R^+} r \frac{u'_0(r)}{u_0(r)} - \lim_{r \rightarrow R^-} r \frac{u'_0(r)}{u_0(r)} = -\lambda(R). \tag{21}$$

From Eqs. (20) and (21) we then get

$$\lambda(R) = \frac{(1/2 - \nu) (R/r_0)^{2\nu} - c(1/2 + \nu)}{(R/r_0)^{2\nu} - c}. \tag{22}$$

We then proceed to the bound-state spectrum ($E = -\frac{\hbar^2 k^2}{2m}$). The radial wave function, the solution to Eqs. (1) together with (19), has the form

$$\begin{aligned}
u(r) &= Cr^{1/2} K_\nu(kr), & r > R, \\
&= C' \sinh(Kr), & r < R,
\end{aligned} \tag{23}$$

where C and C' are normalization constants. Using again the boundary condition (21), we find the following spectral equation:

$$\frac{1}{2} + kR \frac{K'_\nu(kR)}{K_\nu(kR)} - kR \coth(kR) = -\lambda(R). \tag{24}$$

By using formulas (8), we find from Eq. (24) in the limit $kR \ll 1$,

$$\lambda(R) = \frac{\nu + \frac{1}{2} + (\nu - \frac{1}{2}) \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{kR}{2}\right)^{2\nu}}{1 - \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \left(\frac{kR}{2}\right)^{2\nu}}. \tag{25}$$

From Eqs. (22) and (25) we then get

$$k = \frac{2}{r_0} \left[\frac{\Gamma(1+\nu)}{c\Gamma(1-\nu)} \right]^{1/2\nu}. \tag{26}$$

As it is expected, formula (26) is exactly the same expression (10), obtained with a spherical square-well regularization potential, and then it is in complete agreement with the

method of self-adjoint extensions [12]. Recall that the arbitrary scale r_0 , whereby depending on the energy spectrum, must satisfy $R/r_0 \ll 1$ due to the condition $kR \ll 1$, and that the arbitrary constant c must be positive.

As in the previous section, the occurrence of a bound state with formula (26) can be interpreted by the short-range modification of the interaction. In fact, Eq. (22) shows that the strength of the counterterm may take positive values (thus leading to an attractive delta-function potential) even if the inverse-square interaction is zero ($\nu = 1/2$) or repulsive ($1/2 < \nu \leq 1$). Thus, the R-method elucidates the origin of this “counterintuitive” bound state, which can, however, always be suppressed, as in the self-adjoint extension method, by choosing the parameter $c < 0$.

We now consider the particular case $\nu = 0$ ($2m\alpha/\hbar^2 = -1/4$) following the same procedure as in Sec. II.

The zero energy radial wave function $v_0(r)$ now reads

$$\begin{aligned} v_0(r) &= A (r/r_0)^{1/2} [1 + c \ln (r/r_0)], & r > R, \\ &= A (R/r_0)^{-1/2} [1 + c \ln (R/r_0)] (r/r_0), & r < R. \end{aligned} \quad (27)$$

The boundary condition (21) from the delta-function potential at $r = R$ now gives

$$\lambda(R) = \frac{1}{2} - \frac{c}{1 + c \ln (R/r_0)}. \quad (28)$$

The radial wave function for the bound state $v(r)$ has the form

$$\begin{aligned} v(r) &= Cr^{1/2}K_0(kr), & r > R, \\ &= CR^{1/2}\frac{K_0(kR)}{\sinh(KR)}\sinh(kr), & r < R, \end{aligned} \quad (29)$$

where C is a normalization constant and $k^2 = -\frac{2mE}{\hbar^2}$.

Imposing the boundary condition (21) to the solution (29), we now get for $kR \ll 1$,

$$\lambda(R) = -\frac{1}{2} - \frac{1}{\ln(kR)} + kR \coth kR. \quad (30)$$

From Eqs. (28) and (30), we then find

$$k = \frac{1}{r_0} \exp(1/c), \quad (31)$$

with $R/r_0 \ll 1$. We again get a single bound state with exactly the same expression of the binding energy as in the spherical square-well regularization potential [see Eq. (17)].

The latter result confirms the equivalence between the R-method of Beane *et al.* [7] and the method of self-adjoint extensions [12, 13]. On the other hand, our conclusion is in contradiction with the claim of the authors of Refs. [4] advocating the existence of infinitely many bound states for $\nu = 0$.

IV. SUMMARY AND CONCLUSION

We have applied the renormalization method of Beane *et al.* [7] and studied the problem of the singular-inverse square potential α/r^2 in the medium-weak-coupling region, i.e., with $-1/4 \leq 2m\alpha/\hbar^2 \leq 3/4$, by using two regularization potentials, as in Ref. [9]: a spherical square well and a spherical δ shell. As it was expected, the results obtained were independent of the short-range interaction at low energy. We have explicitly shown that, in both of these cases, there exist at most one bound state in the aforementioned range of the coupling constant α , as predicted by the method of self-adjoint extensions. The expression of the bound-state energy has been derived, and we have explicitly shown that this bound state is due to the attractive short-range regularization potential, which corresponds to a given choice of the self-adjoint extension that leads to a bound state even in the absence of a long-range inverse-square interaction ($\alpha = 0$) or in the presence of a repulsive ($0 < 2m\alpha/\hbar^2 \leq 3/4$) long-range interaction. The particular case $2m\alpha/\hbar^2 = -1/4$, which may be relevant to the study of the near horizon structure of black holes [4], has been considered separately. Renormalization leads also to, at most, a single bound state. The latter result is in conflict with the statement of the authors of Refs. [4], who claimed the existence of infinitely many bound states for this critical value of α . On the other hand, our results are in complete agreement with the method of self-adjoint extensions. This leads us to confirm the conclusion of Ref. [8], namely, the equivalence between the R-method of Beane *et al.*, and the method of self-adjoint extensions in the study of singular potentials.

Acknowledgments

The work of D.B is supported by the Algerian Ministry of Higher Education and Scientific Research under the PNR Project No. 8/u18/4327 and the CNEPRU Project No.

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