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# Some Equalities between Elliptic Dilogarithm of 2-Isogenous Elliptic Curves

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**Abstract**

In this paper we establish some equalities between elliptic dilogarithm of the 2-isogenous curves 14A and 14B. This allows us to give a new *exotic* relation for the curve 14B.

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**Keywords:** Elliptic curves, Elliptic dilogarithm, Exotic relations

## 1 Introduction

For some elliptic curves, the elliptic dilogarithm satisfies linear relations other than distribution ones and called *exotic* by Bloch and Grayson [4].

Bloch and Grayson conjectured the following fact.

**Conjecture 1.1** *Suppose that $E(\mathbb{Q})_{tors}$ is cyclic and $d = \#E(\mathbb{Q})_{tors} > 2$. Write $\Sigma$ for the number of fibres of type $I_\nu$ with $\nu \geq 3$ in the Néron model, and suppose $\left[\frac{d-1}{2}\right] - \Sigma > 1$. Then there should be at least $\left[\frac{d-1}{2}\right] - \Sigma - 1$ exotic relations*

$$\sum_{r=1}^{\left[\frac{d-1}{2}\right]} a_r D^E(rP) = 0$$

*where $P$ is a $d$-torsion point and $a_r \in \mathbb{Z}$.*

In particular Bloch and Grayson conjectured for the curve $E_1 = 14A$ the *exotic* relation

$$2D^E(P_1) + 5D^E(2P_1) \stackrel{?}{=} 0$$

where $P_1 = (1, 0)$ is the 6-torsion point of the curve 14A and the notation $A \stackrel{?}{=} B$, means " A is conjectured to be equal to B ", that is A and B are numerically equal to at least 25 decimal places. Recently (2004) this relation was proved by Bertin [2].

By the help of some equalities between elliptic dilogarithm of the 2-isogenous curves 14A and 14B of Cremona's tables [5], we give a new *exotic* relation for the curve 14B.

In [4] only elliptic curves with negative discriminant are considered, so our new *exotic* relation do not appear in the list of Bloch and Grayson since the curve 14B have positive discriminant.

# 2 The elliptic dilogarithm

Let $E$ be an elliptic curve defined over $\mathbb{Q}$.

Throughout this paper, the notation $E = [a_1, a_2, a_3, a_4, a_6]$ means that the elliptic curve $E$ is in the Weierstrass form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

We have two representations for $E(\mathbb{C})$

$$\begin{array}{ccccc} E(\mathbb{C}) & \xrightarrow{\sim} & \mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) & \xrightarrow{\sim} & \mathbb{C}^*/q^{\mathbb{Z}} \\ (\wp(u), \acute{\wp}(u)) & \longrightarrow & u(\mod \Lambda) & \longrightarrow & z = e^{2\pi i u} \end{array}$$

where $\wp$ is the Weierstrass function, $\Lambda = \{1, \tau\}$ the lattice associated to the elliptic curve and $q = e^{2\pi i \tau}$.

**Definition 2.1** *The elliptic dilogarithm $D^E$ [3] is defined by*

$$D^E(P) = \sum_{n=-\infty}^{n=+\infty} D(q^n z)$$

*where $P \in E(\mathbb{C})$ is the image of $z \in \mathbb{C}^*$, $q = e^{2\pi i \tau}$ and $D$ is the Bloch-Wigner dilogarithm,*

$$\begin{array}{llcc} D: & \mathbb{C} \setminus \{0,1\} & \longrightarrow & \mathbb{R} \\ & z & \longmapsto & \Im(Li_2^{[c]}(z) + \log|z| \log^{[c]}(1-z)) \end{array}$$

*where $c$ is a loop from $\frac{1}{2}$ to $z$ in $\mathbb{C} \setminus \{0,1\}$.*

In section 3 we use the two following properties of the Bloch-Wigner dilogarithm

$$D(z) = -D(z^{-1})$$
$$D(z^2) = 2(D(z) + D(-z)).$$

**Remark 2.2** *There is a second representation of the elliptic dilogarithm given by Bloch [3], [9] in terms of Eisenstein-Kronecker series*

$$D^E(P) = \frac{(\Im\tau)^2}{\pi} \Re\Big( \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{\exp(\ 2\pi i(n\xi - m\eta))}{(m\tau + n)^2(m\bar{\tau} + n)}\Big)$$

*where $z = e^{2\pi i u}$ and $u = \xi\tau + \eta$.*

# 3 Equalities between elliptic dilogarithm: a new *exotic* relation

Let $E_1$ be the elliptic curve 14A with equation

$$Y_1^2 + X_1Y_1 + Y_1 = {X_1}^3 - X_1$$

and its 6-torsion point $P_1 = (1, 0)$. Let $E_2$ be the elliptic curve 14B with equation

$$Y_2^2 + X_2Y_2 + Y_2 = {X_2}^3 - 11X_2 + 12$$

and its 6-torsion point $P_2 = (0, 3)$.

**Lemma 3.1** *Using definition 2.1, we find that*

$$D^{E_2}(P_2) = \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(2n+1)\tau_1}), \tag{1}$$

$$D^{E_2}(2P_2) = \sum_{n=-\infty}^{+\infty} D(j e^{2\pi i(2n\tau_1)}), \tag{2}$$

$$D^{E_1}(P_1) = \sum_{n=-\infty}^{+\infty} D(-j e^{2\pi i n\tau_1}), \tag{3}$$

*and*

$$D^{E_1}(2P_1) = \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i n\tau_1}) \tag{4}$$

*where* $j = e^{\frac{2\pi i}{3}}$.

**Proof.** By definition 2.1,

$$D^{E_2}(P_2) = \sum_{n=-\infty}^{+\infty} D(q^n z) = \sum_{n=-\infty}^{+\infty} D(e^{2\pi i n\tau_2} e^{2\pi i u}),$$

using the fact that $\tau_2 = 2\tau_1 + 1$ and $u = \frac{1}{2}\tau_2 + \frac{1}{6}$, we get

$$D^{E_2}(P_2) = \sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(2n+1)\tau_1}).$$

To prove the remaining equalities it suffices to use the fact that $u = \frac{1}{3}$ for $2P_2$, $u = \frac{5}{6}$ for $P_1$ and $u = \frac{2}{3}$ for $2P_1$. ■

Now, we can prove the following theorem.

**Theorem 3.2** *We have the following equalities*

$$\begin{array}{lll} 1) & D^{E_2}(P_2) & = -2D^{E_1}(P_1) + 3D^{E_1}(2P_1) \\ 2) & D^{E_2}(2P_2) & = -2D^{E_1}(P_1) + 2D^{E_1}(2P_1). \end{array}$$

**Proof.** 1) We get from (4)

$$\begin{aligned} D^{E_1}(2P_1) &= \sum_{n=-\infty}^{n=+\infty} D(j^2 e^{2\pi i n\tau_1}) \\ &= \sum_{n=-\infty}^{n=+\infty} D(j^2 e^{2\pi i(2n+1)\tau_1}) + \sum_{n=-\infty}^{n=+\infty} D(j^2 e^{2\pi i(2n)\tau_1}) \\ &= \sum_{n=-\infty}^{n=+\infty} D(j^2 e^{2\pi i(2n+1)\tau_1}) + \sum_{n=-\infty}^{n=+\infty} D((je^{2\pi i n\tau_1})^2), \end{aligned}$$

using (1), (3) and the distribution property of the dilogarithm

$$\sum_{n=-\infty}^{n=+\infty} D((je^{2\pi i n\tau_1})^2) = 2\sum_{n=-\infty}^{+\infty} D(-je^{2\pi i(n\tau_1)}) + 2\sum_{n=-\infty}^{+\infty} D(je^{2\pi i(n\tau_1)})$$

we get,

$$D^{E_1}(2P_1) = D^{E_2}(P_2) + 2D^{E_1}(P_1) + 2\sum_{n=-\infty}^{+\infty} D(je^{2\pi i(n\tau_1)});$$

hence,

$$\begin{aligned} D^{E_1}(2P_1) &= D^{E_2}(P_2) + 2D^{E_1}(P_1) - 2\sum_{n=-\infty}^{+\infty} D((je^{2\pi i(n\tau_1)})^{-1}) \\ &= D^{E_2}(P_2) + 2D^{E_1}(P_1) - 2\sum_{n=-\infty}^{+\infty} D(j^2 e^{-2\pi i(n\tau_1)}) \\ &= D^{E_2}(P_2) + 2D^{E_1}(P_1) - 2\sum_{n=-\infty}^{+\infty} D(j^2 e^{2\pi i(n\tau_1)}). \end{aligned}$$

So, by (4) we get

$$D^{E_1}(2P_1) = D^{E_2}(P_2) + 2D^{E_1}(P_1) - 2D^{E_1}(2P_1).$$

2) We get from (2)

$$\begin{aligned} D^{E_2}(2P_2) &= \sum_{n=-\infty}^{+\infty} D(je^{2\pi i(2n)\tau_1}) = -\sum_{n=-\infty}^{+\infty} D((je^{2\pi i(2n)\tau_1})^{-1}) \\ &= -\sum_{n=-\infty}^{+\infty} D(j^2e^{-2\pi i(2n)\tau_1}) = -\sum_{n=-\infty}^{+\infty} D(j^2e^{2\pi i(2n)\tau_1}) \\ &= -2\sum_{n=-\infty}^{+\infty} D(-je^{2\pi in\tau_1}) - 2\sum_{n=-\infty}^{+\infty} D(je^{2\pi in\tau_1}) \end{aligned}$$

which becomes by (3)

$$D^{E_2}(2P_2) = -2D^{E_1}(P_1) - 2\sum_{n=-\infty}^{+\infty} D(je^{2\pi in\tau_1});$$

hence,

$$\begin{aligned} D^{E_2}(2P_2) &= -2D^{E_1}(P_1) + 2\sum_{n=-\infty}^{+\infty} D((je^{2\pi in\tau_1})^{-1}) \\ &= -2D^{E_1}(P_1) + 2\sum_{n=-\infty}^{+\infty} D(j^2e^{-2\pi in\tau_1}) \\ &= -2D^{E_1}(P_1) + 2\sum_{n=-\infty}^{+\infty} D(j^2e^{2\pi in\tau_1}). \end{aligned}$$

So, by (4) we get

$$D^{E_2}(2P_2) = -2D^{E_1}(P_1) + 2D^{E_1}(2P_1).$$

■

For the elliptic curve 14B, we have $E(\mathbb{Q})_{tors}$ cyclic, $d = \#E(\mathbb{Q})_{tors} = 6$ and $\Sigma = 0$. So the conditions of the conjecture of Bloch and Grayson are satisfied and the curve $14B$ must satisfy an exotic relation. Using the above theorem, we can give a new exotic relation for the curve 14B.

**Corollary 3.3** *The elliptic curve 14B satisfies the following exotic relation*

$$7D^{E_2}(P_2) - 8D^{E_2}(2P_2) = 0.$$

**Proof.** By theorem 3.2 we get the equivalence

$$7D^{E_2}(P_2) - 8D^{E_2}(2P_2) = 0 \Leftrightarrow 2D^{E_1}(P_1) + 5D^{E_1}(2P_1) = 0.$$

The *exotic* relation

$$2D^{E_1}(P_1) + 5D^{E_1}(2P_1) = 0$$

was proved by Bertin [2], so we get

$$7D^{E_2}(P_2) - 8D^{E_2}(2P_2) = 0.$$

■

**Remark 3.4** *We note that in some cases we can see* exotic *relations as relations between elliptic regulators, see Bertin [1, 2] and Touafek [8]. For more details about the elliptic regulator, see Bloch [3] and Rodriguez-Villegas [6, 7]*

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